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Tranched graphs: consequences for topology and dynamics

by

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Abstract. We compare quasi-graphs and generalized $\sin(1/x)$-type continua, which are two classes of continua that generalize topological graphs and contain the Warsaw circle as a nontrivial common element. We show that neither class is a subset of the other, provide some characterizations, and present illustrative examples. We unify both approaches by considering the class of *tranched graphs*, compare it to concepts known from the literature, and describe how the topological structure of its elements restricts possible dynamics.

1. Introduction. In the paper, we study the relationship between *quasi-graphs* and *generalized* $\sin(1/x)$*-type continua*. The two classes were defined independently, nontrivially extending the class of topological graphs. Let us present a brief, informal description of these classes; for a formal definition, the reader is referred to Section 2. If we view topological graphs as arcwise connected unions of arcs, then roughly speaking, by analogy, we can view quasi-graphs as arcwise connected unions of arcs and quasi-arcs. The definition of generalized $\sin(1/x)$-type continua is based on an analogy to objects that generalize the unit interval, the so-called type-λ continua, defined and studied by Kuratowski (see [10, §48, Ch. III, footnote on p. 197]) and subsequently by other mathematicians (see e.g. [16, 18])). In this approach, a topological space generalizes a topological graph if we can define a monotone map onto a topological graph that is 1-1 on a sufficiently large set of points (with additional assumptions). Simple examples suggest that the two definitions may be equivalent under some mild assumptions. For instance, the Warsaw circle (see Figure 1) is both a quasi-graph and a generalized $\sin(1/x)$-type continuum. Motivated by this, we set as the main goal of this

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paper the study of the relationship between the classes of quasi-graphs and of generalized $\sin(1/x)$-type continua.

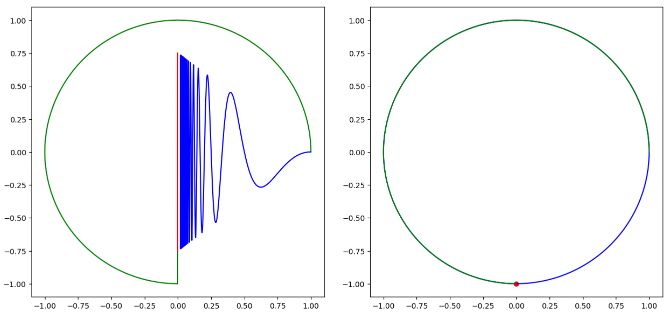

Fig. 1. The Warsaw circle W and its image $\phi(W)$ under the mapping ϕ from Definition 2.4. The points in the topological graph $\phi(W)$ are colored in accordance to their preimage. The oscillatory quasi-arc required by Definition 2.3 is marked in blue.

One can easily find an example of a generalized $\sin(1/x)$-type continuum that is not a quasi-graph, since the definition of a generalized $\sin(1/x)$-type continuum does not demand arcwise connectedness of the space; however, adding arcwise connectedness to the definition still does not lead to equivalence of definitions. The results of the paper include illustrative examples that present the differences between quasi-graphs and generalized $\sin(1/x)$-type continua (see Sections 3 and 4 for more details). We introduce the class of *tranched graphs*, which contains all quasi-graphs and all generalized $\sin(1/x)$-type continua, but is still small enough to provide some concrete characterizations. We also characterize both classes relative to each other, in particular, we try to understand their intersection. For the latter, we find very useful the class of continua known in the literature as Class(W), defined by Lelek in 1972 (see [4] and the summary of classical results in the book of Illanes and Nadler [8, Chs. VIII & IX]).

The main results of the paper can be summarized as follows. In Theorems 3.13 and 3.16 we show that for a quasi-graph X, a sufficient condition to be a generalized $\sin(1/x)$-type continuum is that for any connected component Λ of the union of the limit sets of oscillatory quasi-arcs in X,

(1) there is a quasi-arc $L \subset X$ such that $\omega(L) = \Lambda$,
(2) $\Lambda \in$ Class(W).

Furthermore, (1) is a necessary condition.

In Theorem 4.27 we show that a tranched graph is a quasi-graph if and only if it is arcwise connected and has a finite depth (see Definitions 3.7, 3.11 and 4.18). The result is strengthened by Example 5.1, which shows that the set of assumptions is optimal, by the construction of an arcwise connected generalized $\sin(1/x)$-type continuum with infinite depth. Hence, we get a characterization of generalized $\sin(1/x)$-type continua that are also quasi-graphs, giving us a result in the opposite direction to Theorems 3.13 and 3.16.

The paper is organized as follows. We recall the definitions and basic results in Section 2. The conditions for a quasi-graph to be a generalized $\sin(1/x)$-type continuum are presented in Section 3, while Section 4 contains conditions for the opposite inclusion. Section 5 is devoted to a rigorous construction of an arcwise connected generalized $\sin(1/x)$-type continuum which is a tranched graph of infinite depth. In Section 6 we show how the structure of the spaces considered impacts the dynamics on them.

2. Preliminaries. We denote $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$.

Let X, Y be compact metric spaces. We say that a continuous map $f\colon X \to Y$ is *monotone* if $f^{-1}(y)$ is a connected subset of X for every point $y \in Y$. To distinguish between degenerate and nondegenerate sets $f^{-1}(y)$ induced by a continuous monotone map f in X, we will use the term *fiber* of f for any set $f^{-1}(y)$, $y \in Y$, and reserve the word *tranche* for fibers that are not singletons.

We write $\overline{U}$ for the closure of U, $\operatorname{int} U$ for the interior of U and $\partial U = \overline{U} \setminus \operatorname{int} U$ for the boundary of U. We say that a set is *meager* if it can be written as a countable union of nowhere dense sets, and we call it *residual* if its complement is meager.

For a given metric space (X, d), we denote by d_H the *Hausdorff metric* induced by d on the space 2^X of compact nonempty subsets of X (see [8] for more details).

By the *Hilbert cube* we mean the space $\mathcal{H} = [0, 1]^{\mathbb{N}}$ equipped with the product metric $d(x, y) = \sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|$. By a *continuum* we mean any compact, connected metrizable space. We assume that the reader is familiar with continuum theory (see e.g. [17]), and we only very briefly present some of the concepts to make the paper more self-contained. The hyperspace of all subcontinua of a continuum X is denoted $\mathcal{C}(X) \subset 2^X$. It is well known that it is a closed subset of 2^X and therefore it is also compact [8].

An *arc* is any continuum homeomorphic to the interval $[0, 1]$ with the natural topology, and a (*topological*) *graph* is the union of a finite collection of arcs (called *edges*), intersecting only at their endpoints (called *vertices*). Note that, by definition, the vertices of any edge are distinct points. A topological graph that does not contain any *circle* (a subset homeomorphic to the unit circle with the natural topology induced from the plane) is called a *tree.* Let X be an arcwise connected space and let $x, y \in X$ be distinct. If there is a unique arc $J \subset X$ with endpoints x and y then we denote $[x, y] = J$. It is not difficult to verify that if X is a tree, then $[x, y]$ is defined for any two distinct $x, y \in X$.

A *Peano continuum* is any locally connected continuum (that is, every point has an arbitrarily small open and connected neighborhood). It is well known that being a locally connected continuum is equivalent to being the image of the unit interval $[0,1]$ under some continuous mapping. For a more detailed exposition of Peano continua, the reader is referred to [17].

Denote by S_n a topological graph obtained as the union of n edges that share a common endpoint $s_n \in S_n$, $n \geq 2$. For consistency, denote $S_1 = [0,1]$ and $s_1 = 0$. We say that X is an *n-star centered* at x if there is a homeomorphism $h\colon X \to S_n$ with $h(x) = s_n$.

Definition 2.1. Let X be a nondegenerate, arcwise connected continuum and let $x \in X$.

(i) The *valence* of x is the number

$$\mathrm{val}(x) = \sup_{k\in\mathbb{N}} \{k : \text{there is a } k\text{-star contained in } X \text{ centered at } x\}.$$

(ii) An *endpoint* is any point with valence 1.
(iii) A *branching point* is any point with valence greater than 2.

We write $\mathrm{End}(X)$ and $\mathrm{Br}(X)$ for the sets of endpoints and of branching points of X, respectively.

The original definition of quasi-graphs in [12] states that it is an arcwise connected continuum and there is a natural number N such that for each arcwise connected subset Y, the set $\overline{Y} \setminus Y$ has at most N arcwise connected components. For the purpose of the present paper, we will use the equivalent characterization provided by [12, Theorem 2.24]. Let us first introduce the necessary terminology.

Definition 2.2. Let X be a compact metric space. We say that a subset L of X is a *quasi-arc* with a parameterization φ if the map $\varphi\colon [0,\infty) \to L$ is a continuous bijection. We call the point $\varphi(0)$ the *endpoint* of L. We denote $\omega(L) = \bigcap_{m\geq 0} \overline{\varphi[m,\infty)}$, and say that a quasi-arc is *oscillatory* if $\omega(L)$ has more than one element. It is easy to see that the endpoint and the limit set of a quasi-arc are independent of the parametrization. For consistency, unless stated otherwise, in all figures in this paper, we always depict oscillatory quasi-arcs in blue and their limit sets in red.

In the literature there is a well established notion of *ray*, which is a space L homeomorphic to $\mathbb{R}_+$. A continuum X is called a *compactification* of a ray if it can be represented as the union of a ray R and a continuum P such that $R \cap P = \emptyset$ and $P = \overline{R} \setminus R$. The continuum P is called the *remainder* of the compactification. One can easily verify that every ray is a quasi-arc, and that the class of quasi-arcs is strictly larger as it allows for self-accumulation (we allow $\omega(L) \cap L \neq \emptyset$). For example, both the circle and the Warsaw circle are quasi-arcs, but not rays. Notice that if we view a ray L as a quasi-arc, then the limit set $\omega(L)$ is the remainder of L. For some results about ray compactifications see for example [13, 1].

Nonoscillatory quasi-arcs in X are referred to as 0-*order* quasi-arcs. We say that a quasi-arc L is k-*order* if it contains within its limit set $\omega(L)$ a $(k-1)$-order quasi-arc and the set $\omega(K)$ does not contain any $(k-1)$-order quasi-arcs for any quasi-arc $K \subset \omega(L)$.

If for any $n \in \mathbb{N}$ there exists a sequence $\{L_0, L_1, \dots, L_n\}$ of oscillatory quasi-arcs with $L_0 = L$ and $L_{i+1} \subset \omega(L_i)$, then we say that L is an ∞-*order* oscillatory a quasi-arc. Let $\varphi\colon [0,\infty) \to X$ be a parametrization of a quasi-arc L. If for every $t \in \mathbb{N}$, the quasi-arc $\varphi([t,\infty))$ is not a subset of $\omega(K)$ for any oscillatory quasi-arc K, we say that L is *without ancestors*.

We are now prepared to give a formal definition of quasi-graphs (see [12, Theorem 2.24]).

Definition 2.3. A *quasi-graph* is a continuum X that can be decomposed into a topological graph G and pairwise disjoint oscillatory quasi-arcs $L_1, \dots, L_n$ such that

(i) $X = G \cup \bigcup_{j=1}^{n} L_j$ and $\mathrm{End}(X) \cup \mathrm{Br}(X) \subset G$,
(ii) for each $0 \le i \le n$ we have $L_i \cap G = \{a_i\}$, where a_i is the endpoint of L_i,
(iii) $\omega(L_i) \subset G \cup \bigcup_{j=1}^{i-1} L_j$ for each $0 \le i \le n$,
(iv) if $\omega(L_i) \cap L_j \neq \emptyset$ for some $0 \le i, j \le n$, then $\omega(L_i) \supset L_j$.

We denote $\omega(X) = \bigcup_{i=1}^{n} \omega(L_i)$. We can see that $\omega(X)$ is the union of non-degenerate nowhere dense subcontinua of X, hence it does not depend on the choice of quasi-arcs in the decomposition.

In other words, we can construct every quasi-graph in a finite number of steps as follows. We start with a topological graph and then in each step we add one by one a finite number of oscillatory quasi-arcs (without adding branching points at any step of the construction) in such a way that their limit set is contained in the continuum generated in the previous step.

Now let us define the second class of our interest: generalized $\sin(1/x)$-type continua. The following definition can be found in [6, Section 5] and is a natural generalization of the notion of λ-continuum studied by Kuratowski (see [10, §48, Ch. III, footnote on p. 197]). In the context of λ-continua, the monotone map ϕ in the following definition is sometimes called a *Kuratowski map* (see [15, 16])

Definition 2.4. A continuum X is a *generalized* $\sin(1/x)$-*type continuum* if there exists a topological graph Y and a continuous monotone map $\phi\colon X \to Y$ with the following properties:

(i) $\phi^{-1}(y)$ is nowhere dense in X for any $y \in Y$,
(ii) $\phi^{-1}(D)$ is dense in X, where $D = \{y \in Y : \phi^{-1}(y) \text{ is degenerate}\}$,

(iii) if Y_0 is a subcontinuum of $\phi^{-1}(y)$ and $\epsilon > 0$ then there exists an arc $[a,b] \subset Y$ such that $d_H(Y_0, \phi^{-1}([a,b])) < \epsilon$.

The sets $\phi^{-1}(y)$, $y \in Y$, are called *fibers* (of f) in X, and fibers that are nondegenerate are called *tranches* of X. We will often refer to (iii) as the *approximation property*. As we show later, the decomposition into fibers does not depend on the choice of the graph Y and the map ϕ satisfying the above conditions.

The following is a simple, yet important, observation.

LEMMA 2.5. *Let X, Y be nondegenerate continua and let $\phi\colon X \to Y$ be a continuous monotone map. If the set $\phi^{-1}(D)$ is dense in X, where $D = \{y \in Y : \phi^{-1}(y)$ is degenerate in $X\}$, then $\phi^{-1}(y)$ is nowhere dense in X for every $y \in Y$.*

Proof. The map ϕ is continuous, and hence all fibers $\phi^{-1}(y)$ are closed. Therefore, it is sufficient to show that the fibers have empty interiors. Assume on the contrary that there is a point $y_0 \in Y$ such that $\phi^{-1}(y_0)$ contains an open subset $U \subset \phi^{-1}(y_0)$. Since X is a nondegenerate continuum, it follows that $\phi^{-1}(y_0)$ is a tranche of X. As $\phi^{-1}(D)$ is dense in X, we can find a point y_1 with a degenerate preimage under ϕ such that $\phi^{-1}(y_1) \subset U$. This implies that $\phi^{-1}(y_1) \subset U \subset \phi^{-1}(y_0)$, leading to a contradiction. ∎

REMARK 2.6. By Lemma 2.5 we see that (i) in Definition 2.4 is redundant, so we will omit it in the rest of the paper.

LEMMA 2.7. *Suppose a continuum X, a topological graph Y, and a map $\phi\colon X \to Y$ satisfy* (ii) *of Definition* 2.4. *Then no fiber $\phi^{-1}(y)$ of ϕ is a proper subset of a nowhere dense subcontinuum of X.*

Proof. Suppose that there is a fiber $\phi^{-1}(y)$ for which there is a nowhere dense subcontinuum $U \supset \phi^{-1}(y)$. Denote $V = \phi(U)$. Notice that we can take U such that V is a star with center at y. Choose now $z \in V \setminus \mathrm{End(V)}$ that has a degenerate preimage. As U is nowhere dense, there is a sequence $\{z_n\}_{n=1}^{\infty} \subset X \setminus U$ of elements of singleton fibers converging to $\phi^{-1}(z)$. Clearly each $\phi(z_n)$ is in $X \setminus V$, and by continuity the sequence $\{\phi(z_n)\}_{n=1}^{\infty}$ converges to z. By definition, z is not an endpoint of the star V, which is a contradiction. ∎

From Lemma 2.7 we immediately get the following:

REMARK 2.8. Suppose $\phi_1\colon X \to Y_1$ and $\phi_2\colon X \to Y_2$ satisfy (ii) of Definition 2.4. Then for any maximal (in the sense of inclusion) nowhere dense subcontinuum Λ of X, there are $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $\phi_i^{-1}(y_i) = \Lambda$ for $i = 1, 2$.

3. Characterization of quasi-graphs that are generalized $\sin(1/x)$**-type continua.** In this section we will try to describe which quasi-graphs are generalized $\sin(1/x)$-type continua. The simplest case of the Warsaw circle suggests that the limit sets of quasi-arcs are good candidates for tranches, and this intuition may lead one to a claim that it is a general property (for example [11, see comments after Question 1.1]). Unfortunately, this observation does not generalize to all quasi-graphs as we will show later. Another property that this simple example may suggest is that the limit sets of quasi-arcs satisfy the approximation property (i.e. (iii) of Definition 2.4 holds). This turns out to be false in general as well. To provide a simple example, we will refer to the so-called Class(W).

REMARK 3.1. Over the years, many continua belonging to Class(W) have been discovered, including arc-like continua, nonplanar circle-like continua, hereditarily indecomposable continua, and many others (see [8, Ch. IX, Sec. 67]).

DEFINITION 3.2. A continuum is said to be in Class(W), written $X \in$ Class(W), if for every continuum S and any surjective continuous map $f\colon S \to X$, any subcontinuum of X is the image of a subcontinuum of S.

Here is another characterization of Class(W) (see [4, 19], cf. [8, Ch. VIII, Sec. 35]).

LEMMA 3.3. *A continuum X is an element of* Class(W) *if and only if every compactification Y of $[0,1)$ with remainder X has the property that $\mathcal{C}(Y)$ is a compactification of $\mathcal{C}([0,1))$.*

It follows immediately that if a continuum is not in Class(W) then there is an arc $[0,1)$ and its compactification without the approximation property (hence it is not a generalized $\sin(1/x)$-type continuum). It is well known that n-stars with $n \geq 3$ and circles do not belong to Class(W); to see this, consider for example the continua from Figures 2 and 3. Any monotone map from the definition of $\sin(1/x)$-type continuum has to collapse red subcontinua to a point by Remark 2.8. As a consequence neither of the two examples is a generalized $\sin(1/x)$-type continuum, while the continuum in Figure 3 is a quasi-graph.

LEMMA 3.4. *Suppose that X is a quasi-graph, Y is a topological graph, and $\phi\colon X \to Y$ is continuous and monotone. Suppose that Λ is a connected component of $\omega(X)$. Then $\phi(\Lambda) = \{y\}$ for some $y \in Y$.*

Proof. Suppose on the contrary that there is a quasi-graph X, a graph Y, a continuous monotone map $\phi\colon X \to Y$, and a connected component Λ of $\omega(X) = \bigcup_{i=1}^{n} \omega(L_i)$ with $a, b \in \Lambda$ and $\phi(a) \neq \phi(b)$. First, assume that a and b are in the limit set of the same quasi-arc L_k and fix its parametrization

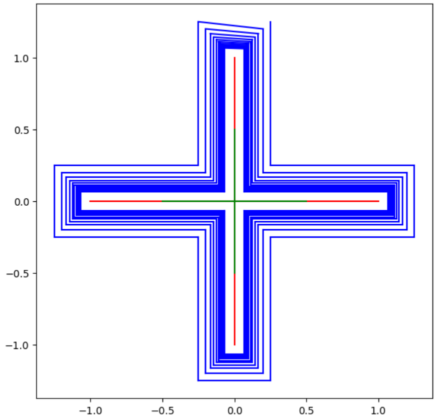

Fig. 2. A quasi-arc with 4-star as the limit set. If a map ϕ collapses the 4-star to a point, then approximation property is violated, e.g. by the subcontinuum marked in green.

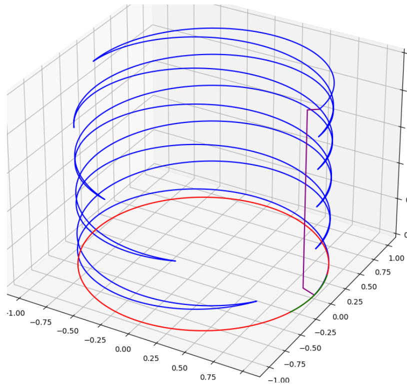

Fig. 3. A quasi-graph whose limit set is circle, but is not a generalized $\sin(1/x)$-type continuum.

$\varphi\colon [0,+\infty) \to L_k$. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset L_k$ be sequences that converge to a and b, respectively. There is a sequence $s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n < \cdots$ such that $\{\varphi([s_n,t_n])\}_{n=1}^{\infty}$ are disjoint arcs, while $\varphi(s_n) = a_n$ and $\varphi(t_n) = b_n$. Then also $J_n = \phi(\varphi([s_n,t_n]))$ is an arc and since $\phi(a) \neq \phi(b)$, there is $\delta > 0$ such that

$$\operatorname{diam}(J_n) \geq d(\phi(a_n),\phi(b_n)) \geq d(\phi(a),\phi(b))/2 > \delta$$

for all sufficiently large n. The arcs J_n cannot be pairwise disjoint for different n, and so there is $x \in \operatorname{int} J_n \cap \operatorname{int} J_m$ for some arbitrarily large $m \neq n$. This means that $\phi^{-1}(x) \cap \varphi([s_n,t_n]) \neq \emptyset$ and $\phi^{-1}(x) \cap \varphi([s_m,t_m]) \neq \emptyset$ while $a_n, a_m \notin \phi^{-1}(x)$ or $b_n, b_m \notin \phi^{-1}(x)$. This immediately implies that $\phi^{-1}(x)$ is not connected, which is a contradiction.

We have shown that there is no quasi-arc such that both a and b belong to its limit set. Let $\Lambda = \omega(L_a) \cup \omega(L_b) \cup \omega(L_{a_1}) \cup \cdots \cup \omega(L_{a_m})$, where $\omega(L_a), \omega(L_b), \omega(L_{a_i})$ are the limit sets of distinct quasi-arcs, with $a \in \omega(L_a)$ and $b \in \omega(L_b)$. We have already proved that $\phi(\omega(L_a)) = \{y\}$ for some $y \in Y$. Suppose now that $\omega(L_a) \cap \omega(L_\xi) \neq \emptyset$ for some $\xi \in \{a_1,\ldots,a_m,b\}$. This means that for some $x_k \in \omega(L_\xi)$ we have $\phi(x_k) = y$, so $\phi(\omega(L_\xi)) = \{y\}$. Then either $\xi = b$, in which case we have a contradiction, or we repeat the above argument until we reach $\omega(L_b)$ (showing $\phi(\omega(L_a)) = \{y\} = \phi(\omega(L_b))$), which must eventually occur as Λ is connected. ∎

The following is a standard but useful tool for the construction of factors (see [9, §19, Theorem 6]). We will use it to project quasi-graphs onto topological graphs.

THEOREM 3.5. *Let D be a decomposition of a Hausdorff space X and let ρ be the corresponding equivalence relation. If D is lower semicontinuous and ρ is closed, then X/ρ equipped with the quotient topology is a Hausdorff space.*

With the above tool at hand, we easily obtain the following result. The details of its (standard) proof are left to the reader.

LEMMA 3.6. *Let X be a quasi-graph. Define an equivalence relation $\sim$ in X by $a \sim b$ if $a = b$ or there exists a connected component Λ of $\omega(X)$ such that $a, b \in \Lambda$. Then $X/\sim$ is a topological graph.*

The continuum depicted in Figure 3 shows that the approximation property of generalized $\sin(1/x)$-type continua in Definition 2.4(iii) sometimes fails for quasi-graphs. Close investigation of these examples shows, however, that the set of singleton fibers is always dense in the quasi-graph, so the condition in Definition 2.4(ii) is satisfied. This motivates the following definition.

DEFINITION 3.7. A continuum X is said to be a *tranched graph* if there is a topological graph Y and a continuous monotone map $\phi\colon X \to Y$ such that $\phi^{-1}(D)$ is dense in X, where $D = \{y \in Y : \phi^{-1}(y) \text{ is degenerate}\}$.

Next we will show that in a tranched graph, the set of singleton fibers is not only dense, but residual, meaning tranched graphs are similar to graphs on a topologically large set.

THEOREM 3.8. *Let X be a tranched graph and let $\phi\colon X \to Y$ be an associated mapping. The set $\phi^{-1}(D)$ is residual in X, where $D = \{y \in Y : \phi^{-1}(y) \text{ is degenerate}\}$.*

Proof. Let $x \in X$ be a point in a singleton fiber, i.e. $\phi^{-1}(\phi(x)) = \{x\}$, and fix any $\varepsilon > 0$. Observe that there is $\delta > 0$ such that if $z \in B(x, \delta)$ then $\operatorname{diam} \phi^{-1}(\phi(z)) < \varepsilon$. Otherwise, there are points w_n, z_n such that $\phi(z_n) = \phi(w_n) \to \phi(x)$ with $d(z_n, w_n) \geq \varepsilon/2$ and as a consequence there are $w \neq z$ with $\phi(x) = \phi(w) = \phi(z)$, which is a contradiction. This shows that the set

$$\mathcal{D}_\epsilon = \{x \in X : \operatorname{diam} \phi^{-1}(\phi(x)) < \varepsilon\}$$

is open and dense.

If we denote by $\mathcal{D}_0$ the set of points belonging to degenerate fibers (i.e. $\mathcal{D}_0 = \phi^{-1}(D)$), then obviously

$$\mathcal{D}_0 = \bigcap_{\epsilon \in \mathbb{Q}_+} \mathcal{D}_\epsilon.$$

As $\mathcal{D}_0$ is residual, this completes the proof. ∎

Now we can prove that properties of a tranched graph do not depend on the choice of map onto a topological graph, as we can go from one to the other by a homeomorphism.

Theorem 3.9. *Assume that X is a tranched graph, let Y_1, Y_2 be two topological graphs and let $\phi_i\colon X \to Y_i$, $i = 1, 2$, be two (possibly different) continuous monotone maps from the definition of tranched graph. Then there is a homeomorphism $\psi\colon Y_1 \to Y_2$ such that $\psi \circ \phi_1 = \phi_2$.*

Proof. By Theorem 3.8, the sets D_1 and D_2 of elements of degenerate fibers of ϕ_1 and ϕ_2 respectively are both residual, hence $D = D_1 \cap D_2 \subset X$ is residual, in particular $\phi_1^{-1}(\phi_1(x)) = \phi_2^{-1}(\phi_2(x)) = \{x\}$ for every $x \in D$.

Let R be the closure of the relation $\{(\phi_1(x), \phi_2(x)) : x \in D\}$ in $Y_1 \times Y_2$. It is clear that Y_1, Y_2 are the projections of R onto the respective coordinates. We claim that R is one-to-one. To see this, fix any $(a, b) \in R$ and let $A = \phi_1^{-1}(a)$ and $B = \phi_2^{-1}(b)$. There is a sequence $\{x_n\}_{n=1}^{\infty} \subset D$ with $x = \lim_{n\to\infty} x_n$ such that $(\phi_1(x_n), \phi_2(x_n)) \to (a, b)$, hence $x \in A \cap B \neq \emptyset$. Then $b \in \phi_2(A)$ and $\phi_2(A)$ must be degenerate as otherwise $D \cap A \neq \emptyset$ and A are nondegenerate, which is impossible. This shows that $A \subset B$ and by a symmetric argument $B \subset A$. Thus R is one-to-one, and hence induces a bijection $\psi\colon Y_1 \to Y_2$.

To see that ψ is continuous, fix any sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ such that $\hat{x}_n \to \hat{x}$ in Y_1. By the previous argument, $\phi_2(\phi_1^{-1}(\hat{x}_n))$ is a single point $\hat{y}_n = \psi(\hat{x}_n)$. Let $\hat{y}$ be any limit point of $\{\hat{y}_n\}_{n=1}^{\infty}$. For every n there is a point $x_n \in D$ such that

$$\operatorname{dist}(x_n, \phi_1^{-1}(\hat{x}_n)) = \operatorname{dist}(x_n, \phi_2^{-1}(\hat{y}_n)) < 1/n$$

and $x_n \to x \in \phi_2(\hat{y})$. But by continuity and uniqueness of the limit we have $\psi_1(x_n) \to \hat{x}$. Since $(\phi_1(x_n), \phi_2(x_n)) \in R$, we have $(\hat{x}, \hat{y}) \in R$, completing the proof. ∎

Remark 3.10. By Theorem 3.9 the decomposition of a tranched graph into fibers, up to homeomorphism, does not depend on the choice of the topological graph Y and the map $\phi\colon X \to Y$. Therefore we can speak about fibers and tranches of X without mentioning ϕ, since it does not lead to ambiguity.

Recall that our first goal is to characterize those generalized $\sin(1/x)$-type continua that are quasi-graphs. The examples presented indicate that we should put some restrictions on the topological structure of the tranches. Definition 3.7, for example, allows a square to be a tranche, which we have to preclude as it cannot be a tranche for any quasi-graph. In Theorem 3.17 we will show that the definition of a generalized $\sin(1/x)$-type continuum does not preclude this either. To deal with this problem, we introduce the following heredity property that imposes some restrictions on tranched graphs.

DEFINITION 3.11. We say that a continuum X is a *tranched graph with hereditary fibers* if X is a tranched graph with an associated continuous monotone map $\phi\colon X \to Y$ onto a topological graph Y such that every fiber $\phi^{-1}(y)$, $y \in Y$, is a singleton or a tranched graph. A continuum X is a *hereditarily tranched graph* if any subcontinuum $A \subset X$ is either a singleton or a tranched graph with hereditary fibers.

It is obvious that all generalized $\sin(1/x)$-type continua are tranched graphs, and the next lemma shows that quasi-graphs also fall into this class. The definition of tranched graph is general enough to cover all quasi-graphs and all generalized $\sin(1/x)$-type continua. Our further goal is to impose additional conditions on tranched graphs to characterize the intersection of the classes of quasi-graphs and generalized $\sin(1/x)$-type continua.

LEMMA 3.12. *Let X be a quasi-graph. Define the equivalence relation $\sim$ on X by $a \sim b$ if $a = b$ or there exists a connected component Λ of $\omega(X)$ such that $a, b \in \Lambda$. Then $X/{\sim}$ is a topological graph and a hereditarily tranched graph. Furthermore, the quotient map $\phi\colon X \to X/{\sim}$ satisfies the condition of Definition* 3.7 *for X.*

Proof. Denote $Y = X/{\sim}$ and observe that by Lemma 3.6, Y is a topological graph. Let $\phi\colon X \to Y$ be the quotient map induced by $\sim$. Tranches in this case are nondegenerate equivalence classes of $\sim$ which are connected components of $\omega(X)$. By Definition 2.3 there are finitely many of them. This means that the set $\phi^{-1}(D)$, where $D = \{y \in Y : \phi^{-1}(y) \text{ is degenerate}\}$, is dense in X.

Recall the original definition of quasi-graphs X from [12] which states that it is an arcwise connected continuum and there is a natural number N such that for each arcwise connected subset Y, the set $\overline{Y} \setminus Y$ has at most N arcwise connected components. It is clear that this property is inherited by any arcwise connected subcontinuum of X, hence any arcwise connected subcontinuum of a quasi-graph is a quasi-graph, which ensures the heredity property for arcwise connected tranches.

Fix a decomposition $X = G \cup \bigcup_{i=1}^{N} L_i$ and any tranche $T = \phi^{-1}(y)$. As $T \subset \omega(X)$ and T is connected by definition, it is the union of finitely many arcwise connected components, $T = G_0 \cup L_{\alpha_1} \cup \cdots \cup L_{\alpha_n}$, $\alpha_i \in \{1, \ldots, N\}$, where $G_0 \subset G$. Namely by Definition 2.3(iv) for every i, either $T \cap L_i = \emptyset$ or $L_i \subset T$. Since endpoints of quasi-arcs are elements of G, we see that $S = G \cup L_{\alpha_1} \cup \cdots \cup L_{\alpha_n}$ is arcwise connected, hence a tranched graph by the previous argument. Then it is easy to see that T in that case is also a tranched graph as $S \setminus T$ is a finite union of topological graphs. We can repeat the above construction (with the analogue of the relation $\sim$) for every tranche T of X.

Suppose now that X_0 is a subcontinuum of X. If X_0 is not a subset of a tranche, then the map ϕ restricted to X_0 satisfies the conditions from Definition 3.7. Suppose now that there is a tranche T_1 in X such that $X_0 \subset T_1$. We know that T_1 is a tranched graph; furthermore, by adding a finite number of arcs we get a quasi-graph with the same set of tranches. It follows that T_1 has hereditary fibers. If X_0 is not nowhere dense in T_1, we get the result as before. If not, it is a subset of a tranche T_2 of T_1, which by our argument is also a tranched graph with hereditary fibers. We continue this process until we find T_m such that X_0 is not nowhere dense in T_m.

By the definition of quasi-graph such T_m must exist. ∎

We have already proved that all quasi-graphs are tranched graphs. We only need to investigate the topological structure of limit sets of oscillatory quasi-arcs in the context of Definition 2.4(iii).

THEOREM 3.13. *Let $X = G \cup \bigcup_{j=1}^{n} L_j$ be a quasi-graph. Assume that for every connected component Λ of $\omega(X)$ the following assertions hold:*

(1) *There is a quasi-arc L in X such that $\omega(L) = \Lambda$*
(2) *The continuum Λ belongs to* Class(W)*.*

Then X is a generalized $\sin(1/x)$*-type continuum.*

Proof. Let $Y = X/{\sim}$, where $\sim$ is defined as in Lemma 3.6, that is, $a \sim b$ if $a = b$ or there exists a connected component Λ of $\omega(X)$ such that $a, b \in \Lambda$. Let $\phi\colon X \to Y$ be the associated quotient map. By Lemma 3.12 we see that X is a tranched graph, so it remains to show that the conditions from Definition 2.4(iii) hold.

Fix any $y \in Y$. If $\phi^{-1}(y) = \{x\}$, there exist neighborhoods U, V of x, y respectively such that $\phi|_U \colon U \to V$ is a homeomorphism. Now suppose that $\phi^{-1}(y)$ is nondegenerate. By assumptions, $\phi^{-1}(y) = \Lambda = \omega(L_c)$ for some quasi-arc L_c. This means that $\Lambda \in$ Class(W) is a compactification of L_c. It follows that for any subcontinuum $Y_0 \subset \Lambda$ there is a sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ of arcs in L_c such that $\{[a_n, b_n]\}_{n=1}^{\infty}$ converges to Y_0 in the Hausdorff metric. One can easily see that this implies the approximation property. ∎

Remark 3.1 recalls most known examples of elements of Class(W). The following may provide another tool for identifying continua in this class.

REMARK 3.14. The result of Grispolakis and Tymchatyn [4, Corollary 3.4] shows that if $Y = L \cup \omega(L)$ for some quasi-arc L then $Y \in$ Class(W) iff $\omega(L) \in$ Class(W). This allows hierarchical constructions of generalized $\sin(1/x)$-type continua.

Unfortunately, Theorem 3.13 provides only a sufficient condition while a complete characterization seems almost an impossible task from the point of view of Theorem 3.17 below. Roughly speaking, it says that any continuum

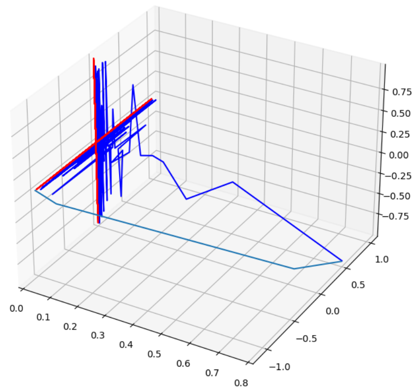

Fig. 4. A quasi-graph which is generalized $\sin(1/x)$-type continuum and contains 4-star as a tranche

can be a tranche of a generalized $\sin(1/x)$-type continuum. For example, we can construct a quasi-graph that has a 4-star as a limit set of its unique oscillatory quasi-arc and satisfies the definition of generalized $\sin(1/x)$-type continuum; see e.g. Figure 4. The only difference compared to Figure 2 is how the quasi-arc approaches its limit set. Then belonging to Class(W) in Theorem 3.13 is only a sufficient condition ensuring that the resulting space is a $\sin(1/x)$-type continuum. On the other hand, the next result shows that condition (1) in Theorem 3.13 is necessary.

LEMMA 3.15. *Let X be a tranched graph with finitely many tranches. Then every tranche is a union of limit sets of finitely many oscillatory quasi-arcs. Moreover, if X is a generalized* $\sin(1/x)$*-type continuum, then every tranche is the limit set of some oscillatory quasi-arc.*

Proof. Let $\phi\colon X \to Y$ be the associated map from Definition 3.7. Fix any tranche $T \subset X$. Denote $y = \phi(T)$. Then there is an open set $U \ni y$ such that $\overline{U}$ contains at most one branching point. As there are finitely many tranches, we can pick $\epsilon > 0$ such that $B = B(T, \epsilon)$ does not intersect any tranches other than T and $\phi(B) \subset U$.

Now let $n = \mathrm{val}(y)$ and choose a closed set $V \subset Y$ such that $U \supset V = E_1 \cup \cdots \cup E_n$ is an n-star with center y and edges E_i. By continuity of ϕ and the fact that it is one-to-one on $B \setminus T$, the sets $T_i = \phi^{-1}(E_i \setminus \{y\})$ are quasi-arcs for $i = 1, \ldots, n$ and $\bigcup_{i=1}^n \omega(T_i) \subset T$. Assume now there is $x \in T$ that is not an element of the limit set of a quasi-arc. Then, by the definition of topological limit, there is a ball centered at x that does not intersect any quasi-arc. Choosing the radius of this ball to be smaller than ϵ if necessary, we get an open ball contained in T, contradicting the assumption that fibers are nowhere dense in X. Summing up, we get $T = \bigcup_{i=1}^n \omega(T_i)$, which proves the first statement of the theorem.

Suppose now that X is a generalized $\sin(1/x)$-type continuum but there is a tranche $\Lambda \subset X$ that is not the limit set of any quasi-arc, i.e. $\Lambda \neq \omega(L_k)$ for any oscillatory quasi-arc $L_k \subset X$. By the previous argument we know that $\Lambda = \bigcup_{i\in K} \omega(L_i)$, where K is the set of indices of quasi-arcs whose limit sets are subsets of Λ. Each $\omega(L_k)$ is connected, thus K is well defined. Denote $\delta = \min_{k\in K} \operatorname{diam}(\Lambda \setminus \omega(L_k)) > 0$.

First we claim that if $Y_0 \subsetneq \Lambda$ is a proper subcontinuum of Λ then for $\epsilon > 0$ small enough, any arc $[a,b]$ such that $d_H(Y_0, \phi^{-1}([a,b])) < \epsilon$ may intersect only one quasi-arc. To see this, suppose that $[a,b] \subset Y$ is an arc with the desired property such that $\phi^{-1}(a)$ and $\phi^{-1}(b)$ are singletons and elements of different quasi-arcs. Therefore, since ϵ is small, we must have $\Lambda \cap \phi^{-1}([a,b]) \neq \emptyset$ and consequently $\Lambda \subset \phi^{-1}([a,b])$. This implies that

$$\epsilon > d_H(\phi^{-1}([a,b]), Y_0) \geq d_H(\Lambda, Y_0) > 0$$

But ϵ can be arbitrarily small, hence we may assume that $\epsilon < d_H(Y_0, \Lambda)$, which is a contradiction.

Now choose a subcontinuum $Y_0 \subset \Lambda$ such that $0 < \operatorname{diam}(\Lambda \setminus Y_0) < \delta$ and fix a small $\epsilon > 0$ provided by the claim above and assume that $\epsilon < \delta$. Since X is a generalized $\sin(1/x)$-type continuum, there is an arc $[a,b] \subset Y$ such that $d_H(\phi^{-1}([a,b]), Y_0) < \epsilon$. By the choice of ϵ we cannot have $Y_0 \subset \phi^{-1}([a,b])$, thus we can assume that $[a,b] \subset L$ for some quasi-arc L with $\omega(L) \subset \Lambda$. Letting $\varepsilon \to 0$ and using the pigeon-hole principle, we find that $Y_0 \subset \omega(L)$ for some quasi-arc L and therefore $\operatorname{diam}(\Lambda \setminus Y_0) \geq \operatorname{diam}(\Lambda \setminus \omega(L))$. It follows that

$$\delta > \operatorname{diam}(\Lambda \setminus Y_0) \geq \operatorname{diam}(\Lambda \setminus \omega(L)) \geq \delta,$$

which is a contradiction. ∎

THEOREM 3.16. *Let X be a quasi-graph that is a generalized* $\sin(1/x)$-*type continuum. Then for every connected component Λ of $\omega(X)$ there is a quasi-arc $L \subset X$ such that $\omega(L) = \Lambda$.*

Proof. By Lemma 3.12 every quasi-graph is a hereditarily tranched graph with tranches being connected components of $\omega(X)$, in particular with finitely many tranches. Lemma 3.15 completes the proof. ∎

In the introduction, we were trying to convince the reader that the main goal behind the definition of a generalized $\sin(1/x)$-type continuum was to obtain a nice generalization of topological graphs. In a sense $\phi\colon X \to Y$ should maintain the structure of X "similar" to Y. Surprisingly, there is no direct restriction on the structure of a single tranche in a generalized $\sin(1/x)$-type continuum. It can be any continuum, extending the class of generalized $\sin(1/x)$-type continua much beyond the initial intuition. The following result is folklore and is not difficult to prove. The details are left to the reader (cf. the arguments in [19]).

THEOREM 3.17. *Suppose X is a continuum. Then there exists an oscillatory quasi-arc L with $\omega(L)$ homeomorphic to X and such that $Z = \omega(L) \cup L$ is a generalized* $\sin(1/x)$*-type continuum.*

REMARK 3.18. Suppose $X = G \cup \bigcup_{j=1}^{n} L_j$ is a quasi-graph. Then there is an oscillatory quasi-arc L_{n+1} such that $\omega(L_{n+1})$ is homeomorphic to $G \cup \bigcup_{j=1}^{n} L_j$ and $Z = L_{n+1} \cup \omega(L_{n+1})$ is a quasi-graph and a generalized $\sin(1/x)$-type continuum. Furthermore, if $X \subset \mathcal{H} \times \{0\} \times \{0\}$ then L_{n+1} can be constructed in such a way that $Z = G \cup \bigcup_{j=1}^{n+1} L_j$ (i.e. we have an exact representation without passing through a homeomorphism). Namely, we can use the first two coordinates to define a spiral compactified by $\{0\} \times \{0\}$ and use its consecutive segments as parameterizations of arcs approximating X in $\mathcal{H}$. This way we obtain an oscillatory quasi-arc with remainder X.

So far, we provided necessary conditions ensuring that a given quasi-graph is a generalized $\sin(1/x)$-type continuum (Theorem 3.13) and that one of the conditions is necessary (Theorem 3.16). Lastly, we have proved that any continuum can be a tranche (Theorem 3.17). Taking these results all together, it seems that the characterization in full generality when a quasi-graph is also a generalized $\sin(1/x)$-type continuum is out of reach.

4. Characterization of generalized $\sin(1/x)$-type continua that are quasi-graphs. Recall that quasi-graphs which are generalized $\sin(1/x)$-type continua must have a finite number of tranches, because their tranches are connected components of limit sets of oscillatory quasi-arcs, and quasi-graphs have a finite number of those. It may happen, however, that a generalized $\sin(1/x)$-type continuum has uncountably many tranches (see [6, Example 26]). As a complement to these results, we will construct a generalized $\sin(1/x)$-type continuum with an infinite "depth" (see Lemma 4.6).

LEMMA 4.1. *Let X be a generalized* $\sin(1/x)$*-type continuum. Then X contains at most countably many oscillatory quasi-arcs without ancestors.*

Proof. Suppose X has an uncountable family $\{L_\alpha\}_{\alpha \in A}$ of pairwise disjoint oscillatory quasi-arcs without ancestors. Fix a parameterization $\phi_\alpha \colon [0, +\infty) \to L_\alpha \subset X$ of L_α. Denote $\mathring{L}_\alpha = L_\alpha \setminus \phi_\alpha(0)$. Let $\phi \colon X \to Y$ be the map provided by the definition of a generalized $\sin(1/x)$-type continuum. As points with nondegenerate preimage have to be nowhere dense in Y, ϕ is a bijection on each $\mathring{L}_\alpha$, hence disjoint quasi-arcs do not map by ϕ onto the same arcs in Y. The sets $\mathring{L}_\alpha$ may intersect, but it is not hard to see that the intersections are allowed only at the branching points of Y, so there are only finitely many L_α such that $\phi(L_\alpha)$ contains a branching point. By removing the indices of those arcs from A (should there be any) we get a family of disjoint open arcs in G with cardinality no smaller than that of A (in particular uncountable). But

Y is a topological graph, and hence it does not allow an uncountable family of open connected disjoint subsets. This contradiction completes the proof. ∎

REMARK 4.2. A generalized $\sin(1/x)$-type continuum can contain only countably many oscillatory quasi-arcs but may contain uncountably many tranches. This means that the situation where a tranche is the limit set of an oscillatory quasi-arc (as in the Warsaw circle) is to some extent special.

In [6, Example 26] the authors provided an example of a generalized $\sin(1/x)$-type continuum X with a map $\phi: X \to [0,1]$ such that there is a Cantor set $Q \subset [0,1]$ with $\phi^{-1}(y)$ nondegenerate for every $y \in Q$. It is an example of the situation described in Remark 4.2.

Later, in Example 4.8 we show that an even stronger extreme case is possible. We will construct a generalized $\sin(1/x)$-type continuum X with a dense set of tranches and without an oscillatory quasi-arc (in fact, X does not contain any arcs).

LEMMA 4.3. *Suppose that X is a tranched graph and $L \subset X$ is an oscillatory quasi-arc. Then $\omega(L)$ is a subset of a tranche.*

Proof. As $\omega(L)$ is a nowhere dense subcontinuum of X, the result follows from Remark 2.8. ∎

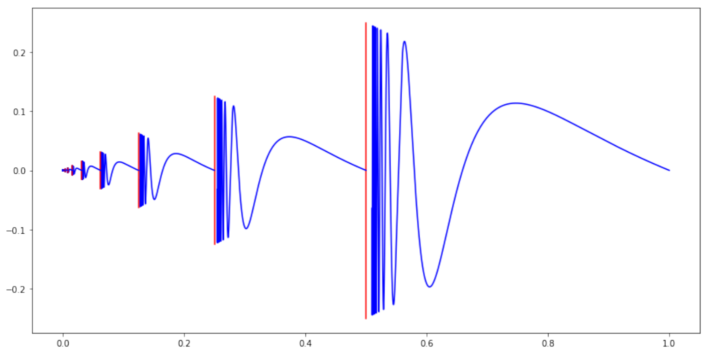

Fig. 5. An example of a generalized $\sin(1/x)$-type continuum whose set of tranches is not closed

For the construction in Example 4.4 below we will use a continuum homeomorphic to the classical $\sin(1/x)$-continuum. However, note that we slightly modified the classical geometry of this object. Although we will still require that its limit set is $\{0\} \times [0,1]$, we also demand that no point of the associated oscillatory quasi-arc intersects $[0,1] \times \{0\}$ or intersects $[0,1] \times \{1\}$ only at its endpoint (see Figure 6). Clearly, we can view this quasi-arc as the graph of a continuous map from the set $(0,1]$ onto itself, allowing the following inductive procedure.

EXAMPLE 4.4. Let $f\colon (0,1] \to (0,1]$ be a continuous surjective mapping such that $f(x) = 1$ if and only if $x = 1$, $f(x)$ approaches 0 like the classical $\sin(1/x)$-continuum and $f(x) \neq 0$ for any $x \in (0,1]$; see Figure 6. Strictly speaking, we put $f(t) = 0.5((1-t)\sin(1/t)+1)$ for $t \leq 0.7$ and for $t \geq 0.7$, the map f is affine with $f(1) = 1$.

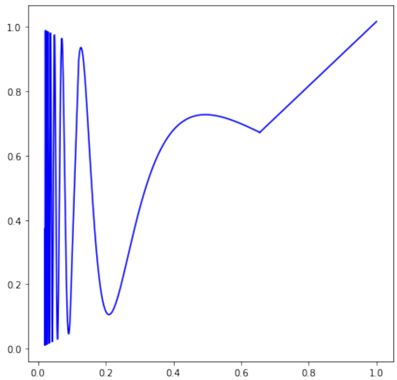

Fig. 6. The map $f\colon (0,1] \to (0,1]$ from Example 4.4

Let us now consider the following continua:

$$
\begin{aligned}
A_0 &= \{(x,0,0,\ldots) : x \in (0,1]\} \cup \{0\}^\infty, \\
A_1 &= \{(x,f(x),0,\ldots) : x \in (0,1]\} \\
&\quad \cup \{(0,x,0,\ldots) : x \in (0,1]\} \cup \{0\}^\infty, \\
&\ \vdots \\
A_n &= \{(x,f(x),\ldots,f^n(x),0,\ldots) : x \in (0,1]\} \cup \theta(A_{n-1}), \\
&\ \vdots
\end{aligned}
\tag{4.1}
$$

where $\theta\colon [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is the right shift defined by $\theta((x_0,x_1,\ldots)) = (0,x_0,x_1,\ldots)$.

It is easy to see that the sequence $\{A_n\}_{n=1}^\infty$ converges, as the difference between the nth and $(n+1)$th elements only appears in the $(n+1)th$ coordinate. Now let $A = \lim_{n\to\infty} A_n$, or equivalently

$$
A = \bigcup_{n=0}^{\infty} \theta^n(\{x,f(x),\ldots,f^n(x),\ldots) : x \in (0,1]\}) \cup \{0\}^\infty. \tag{4.2}
$$

In what follows, we will take a closer look at the properties of the set A. It will provide us with an intuition before proceeding with more complicated examples of similar flavor.

LEMMA 4.5. *Each of the continua A_n defined by* (4.1) *is a generalized* $\sin(1/x)$*-type continuum and is arc-like.*

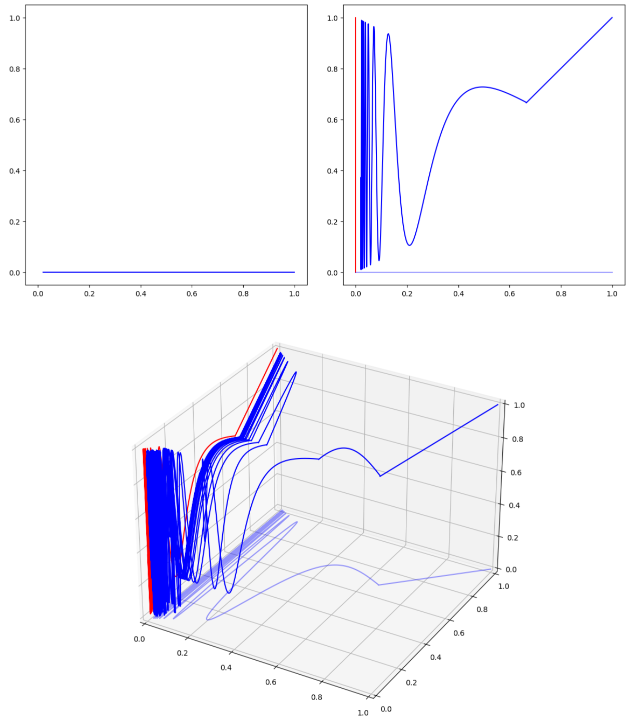

Fig. 7. Projections of the continua A_0, A_1, A_2 defined by (4.1)

Proof. It is enough to use Remarks 3.1 and 3.14. The continuum A_1 is an interval, hence is arc-like and an element of Class(W). Observe that A_{n+1} is the closure of an oscillatory quasi-arc L_{n+1} with $\omega(L_{n+1}) = \theta(A_n)$. Since θ is a homeomorphism between A_n and $\theta(A_n)$, we see that $\theta(A_n)$ is arc-like, and so $A_{n+1} = L_{n+1} \cup \omega(L_{n+1})$ is a generalized $\sin(1/x)$-type continuum, as $\omega(L_{n+1}) \in$ Class(W). It is elementary to check that since $\theta(A_{n+1})$ is arc-like, so is A_{n+1}. ∎

LEMMA 4.6. *The set A defined by* (4.2) *is a generalized* $\sin(1/x)$*-type continuum.*

Proof. It is clear that A is compact. Suppose it is not connected. Then there are closed and disjoint sets U, V such that $A = U \cup V$ and so there is a coordinate n such that the projections U_n and V_n are also disjoint. The projection of A onto the nth coordinate is the set

$$\bigcup_{i=0}^{n} \{f^i(x) : x \in (0,1]\} \cup \{0\} = [0,1] = U_n \cup V_n,$$

which is a contradiction.

Let $Y = [0,1]$ and let $\phi\colon A \to Y$ be the projection onto the first coordinate. Observe that $\phi^{-1}(\{0\}) = \{x \in A : x_0 = 0\} = \theta(A)$, hence it is a compact connected set. For $x \in (0,1]$ each fiber $\phi^{-1}(x)$ is a singleton. This shows that ϕ is a monotone map.

Let Z be a subcontinuum of $\phi^{-1}(0)$. Note that the projection of A onto the first $n+1$ coordinates is the same as the projection of A_n onto these coordinates. Take n such that if y, z satisfy $y_i = z_i$ for $i = 0, \ldots, n$ then $d(y,z) < \varepsilon/4$. Let Z_n be the projection of Z onto the first $n+1$ coordinates with 0 on all coordinates with index $i > n$. In other words, Z_n is the projection of Z into A_n. Let $\varphi_n(x) = (x, f(x), \ldots, f^n(x), 0, 0, \ldots) \in A_n$ and $\varphi(x) = (x, f(x), \ldots, f^n(x), \ldots) \in A$, defined for $x \in [0,1)$, be oscillatory quasi-arcs. By Lemma 4.5 there are $a, b \in (0,1]$ such that $d_H(\varphi_n([a,b]), Z_n) < \varepsilon/4$. But $d_H(\varphi_n([a,b]), \varphi([a,b])) < \varepsilon/4$, hence $d_H(\varphi([a,b]), Z) < 3\varepsilon/4 < \varepsilon$, completing the proof. ∎

Let us observe that A has a kind of fractal self-similar structure. Strictly speaking, we can construct a sequence $A = X_0 \supset X_1 \supset \cdots \supset X_n$ of any finite length such that X_{k+1} is a tranche of X_k for $k = 0, \ldots, n-1$ and all of the spaces X_k are homeomorphic to A. The construction in Example 4.4 is also a natural way of constructing an ∞-order oscillatory quasi-arc. This gives us another difference between quasi-graphs and generalized $\sin(1/x)$-type continua.

Remark 4.7. In general, generalized $\sin(1/x)$-type continua may contain ∞-order oscillatory quasi-arcs as subsets.

To construct the continuum A defined by 4.2 we set a particular geometric representation of the $\sin(1/x)$-continuum, whose only intersection with the line $[0,1]\times\{0\}$ was allowed for the limit set of the $\sin(1/x)$ curve. This allowed us to lift this continuum consecutively to higher dimensions, keeping only one tranche each time, hence maintaining a general structure of oscillatory quasi-arc.

In Example 4.8 we are going to give another geometric representation of a $\sin(1/x)$-type continuum, but this time, for symmetry in the construction, we use a two-sided version of the $\sin(1/x)$-continuum (see Figure 8). If we discard the limit sets of the $\sin(1/x)$-curve, the remaining set can be parameterized as the graph of a function. The main difficulty in repeating this construction (and in providing an accessible description of the resulting space) is caused by the placement of the curve in space. Namely, several points map to 0 or 1, and so we cannot iterate the map on them any further (in contrast to Example 4.4 where there was only one problematic point). This significantly increases the complexity of the construction (and the resulting space). In contrast to Example 4.4, instead of maintaining one tranche, in each step of the construction, we produce infinitely many new tranches.

EXAMPLE 4.8. First define an auxiliary sequence by setting $a_{-k} = \frac{1}{(k+2)^2}$ for $k \in \mathbb{N}$ and $a_k = 1 - \frac{1}{(k+2)^2}$ for $k > 0$. Notice that the intervals $I_n = [a_n, a_{n+1}]$ give a decomposition of $(0,1)$, $|I_n| \leq 1/2$ for all $n \in \mathbb{Z}$ and $|I_n| = 1/2$ if and only if $n = 0$. On each interval I_n we plot a tent map with slope $\lambda_n = 2\frac{1}{|I_n|}$; notice that $\lambda_n \geq 4$. Denote by X the closure of the graphs of those tent maps (see Figure 8).

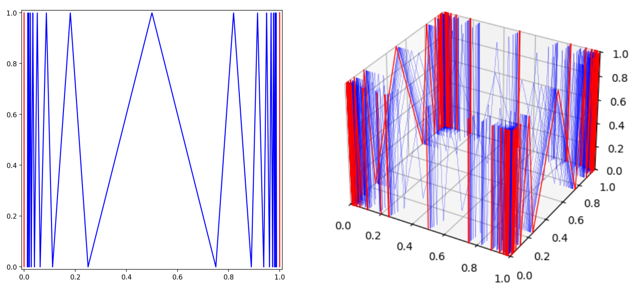

Fig. 8. The continuum X and a projection of the continuum X_2 from Example 4.8. Points where X_2 is not locally arcwise connected are marked red. Roughly speaking, they appear at the faces of the boundary cube; on two faces these sets are copy of X; on the other two adjacent faces they appear as vertical lines exactly at the extrema of the blue curve defining X.

It is easy to check that X is a generalized $\sin(1/x)$-type continuum. We construct the following sequence of continua embedded in the Hilbert cube $\mathcal{H} = [0,1]^{\mathbb{N}}$:

$$X_0 = \{(x_0, 0, 0, \dots) : x_0 \in [0,1]\},$$
$$X_1 = \{(x_0, x_1, 0, \dots) : x_0 \in [0,1], (x_0, x_1) \in X\},$$
$$\vdots$$
$$X_n = \{(x_0, x_1, \dots, x_n, 0, \dots) : x_0 \in [0,1], \forall i = 1, \dots, n \ \ (x_{i-1}, x_i) \in X, \}$$
$$\vdots$$

The ultimate goal of our construction is to prove that the sequence $\{X_n\}_{n=1}^{\infty}$ converges in the Hausdorff metric to a generalized $\sin(1/x)$-type continuum. Before we proceed further, we describe the topological structure of X_n. Let $\psi \colon X \to [0,1]$ and $\psi_n \colon X_n \to [0,1]$ be the projections onto the first coordinate. The following lemma describes the subcontinua of X_n:

LEMMA 4.9. *Let $Z \subset X_n$ be a subcontinuum with $\psi_n(Z) = y$ for some $y \in [0,1]$. Then $Z = \{y\} \times Z_0$ and Z_0 is a subcontinuum of X_{n-1}.*

Proof. As $\psi_n\colon X_n \to [0,1]$ is a projection, obviously $Z = \{y\} \times Z_0$ for some set Z_0. Denote by σ the standard left shift, meaning $\sigma((x_0, x_1, \ldots)) = (x_1, x_2, \ldots)$. Then $Z_0 = \sigma(Z)$ and so it is a continuum, since σ is a continuous map. But the relation X is surjective, and hence by definition $\sigma(X_n) = X_{n-1}$, completing the proof. ∎

Now we are ready to prove the following:

LEMMA 4.10. *Each continuum X_k, $k = 0, 1, \ldots$, is a generalized* $\sin(1/x)$-*type continuum with the associated map $\psi_k\colon X_k \to [0,1]$* (*that is, ψ_k satisfies the definition requirements*).

Proof. We will proceed by induction. By the definition, X_0 is an arc and X_1 is homeomorphic to X, so both are generalized $\sin(1/x)$-type continua. As we noticed before, X can be viewed as the closure of the graph of a function, say $f\colon (0,1) \to [0,1]$.

Now assume we have already proved that X_{k-1} is a generalized $\sin(1/x)$-type continuum for some $k \geq 2$. Choose a point $x_0 \in [0,1]$ such that $\psi_k^{-1}(x_0)$ is nondegenerate and fix any point $x = (x_0, \ldots, x_k, 0, \ldots) \in \psi_k^{-1}(x_0) \subset X_k$. We claim that for every $\epsilon > 0$ there is an element $x_\epsilon \in X_k$ defining a singleton fiber and such that $d(x, x_\epsilon) < \epsilon$.

Assume that $x_0 = 0$ and $x_1, \ldots, x_k \notin \{0,1\}$. Then using the fact that X is a generalized $\sin(1/x)$-type continuum, for every $\epsilon > 0$ there is a point $(z, x_1) \in X$ with $0 < z < \epsilon$. By our construction $x_\epsilon = (z, x_1, \ldots, x_k, 0, \ldots) \in X_k$ and obviously $d(x, x_\epsilon) < \epsilon$. For $x_0 = 1$ the process is completely analogous. If there were positions $x_i \in \{0,1\}$ we can perform the above construction of x_ε in steps, starting with largest $i \leq k$ such that $x_i \in \{0,1\}$ and perform the above modification starting with this position, next move to smaller i, eventually finishing at x_0.

Suppose now $x_0 \notin \{0,1\}$, and choose $\delta < \epsilon/2$ such that $J = (x_0 - \delta, x_0 + \delta) \subset (0,1)$ and $f(J) \subset [x_1 - \epsilon/2, x_1 + \epsilon/2]$. By the induction hypothesis there is a point $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_k, 0, 0, \ldots)$ such that $d((x_1, \ldots, x_k, 0, \ldots), \tilde{x}) < \epsilon/2$ and $\tilde{x}_1 \in f(J)$, which is possible since $f(J) \ni x_1$ is an open set for small δ. We choose $\tilde{x}_0 \in J$ such that $f(\tilde{x}_0) = \tilde{x}_1$ and we set $x_\epsilon = (\tilde{x}_0, \ldots, \tilde{x}_k)$. It follows easily that $d(x, x_\epsilon) < \epsilon$. We see that the set of degenerate fibers is dense in X_k, so the claim is proved.

Next we are going to prove the approximation property. First we claim that if $Z \subset X_k$ is a continuum such that $\psi_k(Z) = [a,b]$ for a nondegenerate interval $[a,b] \subset [0,1]$, then $Z = \psi_k^{-1}([a,b])$. Denote by $D_k \subset [a,b]$ the set of points with degenerate preimages under ψ_k. We have already proved that points from singleton fibers are dense in X_k, and the argument easily leads to $\overline{\psi_k^{-1}(D)} = \psi_k^{-1}([a,b])$. On the other hand, points in $\psi_k^{-1}(D)$ map injectively to $[a,b]$, meaning $\psi_k^{-1}(D) \subset Z$, and so Z is dense in $\psi_k^{-1}([a,b])$. But Z is closed, so $Z = \psi_k^{-1}([a,b])$ proving the claim.

Fix any $y \in [0,1]$ and $\epsilon > 0$. If $\psi_k^{-1}(y)$ is a singleton, the approximation property is trivially satisfied. Suppose that $\psi_k^{-1}(y)$ is nondegenerate and fix any subcontinuum Y_0 of $\psi_k^{-1}(y)$. Lemma 4.9 implies that $Y_0 = \{y\} \times Z_0$ for some subcontinuum Z_0 of X_{k-1}. Now either $\psi_{k-1}(Z_0) = [a,b]$ for $a \neq b$, or Z_0 is a subset of a tranche in X_{k-1}. Assume the former. Choose $[\alpha,\beta] \subset [0,1]$ such that $\operatorname{dist}(y,[\alpha,\beta]) < \epsilon$ and $R([\alpha,\beta]) = [a,b]$, where $R(J) = \{z \in [0,1] : (x,z) \in X, x \in J\}$. Then as shown before, there is a unique continuum $A = \phi_k^{-1}([\alpha,\beta])$ that projects to $[\alpha,\beta]$, and so $\sigma(A) = Z_0$, σ being the left shift, defined as in Lemma 4.9. That gives

$$d_H(Y_0, \psi_k^{-1}([\alpha,\beta])) = \operatorname{dist}(y,[\alpha,\beta]) < \epsilon.$$

Suppose now that Z_0 is a subset of a tranche in X_{k-1}. By the induction hypothesis, there is $[a,b] \subset [0,1]$ with

$$d_H(\psi_{k-1}^{-1}([a,b]), Z_0) < \epsilon/2.$$

Again, we choose $[\alpha,\beta] \subset [0,1]$ such that $\operatorname{dist}(y,[\alpha,\beta]) < \epsilon/2$ and $R([\alpha,\beta]) = [a,b]$, giving

$$d_H(Y_0, \psi_k^{-1}([\alpha,\beta])) < \epsilon. \ \blacksquare$$

It is easy to see that the sequence $\{X_n\}_{n=1}^{\infty}$ defined in Example 4.8 converges, as the changes we make occur in higher dimensions in each step. Denote $\widehat{X} = \lim_{n\to\infty} X_n$ and let $\hat{\psi}\colon \widehat{X} \to [0,1]$ be the projection onto the first coordinate. The construction of $\widehat{X}$ is similar to that of an inverse limit; it and is sometimes called the infinite Mahavier product of X, and has generated some attention recently (see [2]). We are going to show that $\widehat{X}$ is a generalized $\sin(1/x)$-type continuum. The first step is to show that all fibers are nowhere dense, which is (i) in Definition 2.4 and will be used to prove point (ii) in that definition.

LEMMA 4.11. *The fiber $\hat{\phi}^{-1}(y)$ is nowhere dense in $\hat{X}$ for any $y \in [0,1]$.*

Proof. If the fiber is a singleton, the result is obvious. Assume otherwise and suppose that there is an open set $U \subset \hat{\phi}^{-1}(y)$. There is n such that U has a nondegenerate projection onto the nth coordinate. But then $\psi_n^{-1}(y)$ has nonempty interior in X_n, which contradicts Lemma 4.10. ■

LEMMA 4.12. *Let $N = [0,1] \setminus D$, where $D \subset [0,1]$ is the set of points with a degenerate preimage. The set $\hat{\psi}^{-1}(N)$ is dense in $\widehat{X}$, countable, and meager. As a consequence, $\hat{\psi}^{-1}(D)$ is residual.*

Proof. Observe that X_1 has two tranches, the ψ_1-preimages of 0 and 1. It follows that the distance between the projections of the tranches by ψ_1 is 1. By our construction, in X_2 the longest open subintervals of $[0,1]$ with all points having degenerate preimages under ψ_2 are $\big(a_0, \frac{a_0+a_1}{2}\big)$ and $\big(\frac{a_0+a_1}{2}, a_1\big)$, both of length $\frac{1}{\lambda_0}$, which are pieces of monotonicity of the tent map with the smallest possible slope used to define X. These intervals get stretched by a factor of λ_0, so it is easy to see that the longest interval without a degenerate preimage under ψ_3 has length $\frac{1}{\lambda_0}\frac{1}{\lambda_0} = \frac{1}{\lambda_0^2}$. Continuing inductively we find that the longest interval without a nondegenerate ψ_{n+1}-preimage has length $\frac{1}{\lambda_0^n}$. By passing to the limit, we deduce that for any $y \in [0,1]$ and any $\epsilon > 0$ the set $(y-\epsilon, y+\epsilon) \cap N$ is nonempty and so N is dense in $[0,1]$.

We are going to repeat the argument from the proof of Lemma 4.10 to show that $\hat{\psi}^{-1}(N)$ is also dense. Take any point y with a degenerate preimage. By the density of N, there is a sequence $\{y_n\}_{n=1}^{\infty} \subset N$ such that $y_n \to y$. For each i fix $x_i \in \hat{\psi}^{-1}(y_i)$ and assume by compactness that the limit $x_n \to x$ exists. But then $\hat{\psi}(x_n) = y_n \to y = \hat{\psi}(x)$ and therefore $\hat{\psi}^{-1}(y) = \{x\}$, so indeed $\hat{\psi}^{-1}(N)$ is dense in X as claimed.

The set of points with a nondegenerate preimage under ψ_k is countable for all k, and N is a countable union of such sets, therefore N is countable.

By Lemma 4.11 the set $\hat{\psi}^{-1}(N)$ is meager. As $\hat{\psi}^{-1}(D)$ is its complement in $\widehat{X}$, $\hat{\psi}^{-1}(D)$ is a residual set. ∎

The tranches of the continua X_k we have constructed are homeomorphic to X_{k-1}. This extends to $\widehat{X}$.

LEMMA 4.13. *Each fiber $\hat{\psi}^{-1}(y) \subset \widehat{X}$ either is a singleton or is homeomorphic to $\widehat{X}$.*

Proof. Let $y \in [0,1]$ be such that $\hat{\psi}^{-1}(y)$ is not a singleton. This means that there is $k \in \mathbb{N}$ such that $\psi_k^{-1}(y)$ is a tranche of X_k. Fix the smallest k with this property, and note that $\psi_{k-1}^{-1}(y)$ is not a tranche of X_{k-1}. Therefore the first $k-1$ coordinates of any element in $\hat{\psi}^{-1}(y)$ are uniquely determined and belong to $(0,1)$. Using the symmetry of X, without loss of generality we may assume that the kth coordinate of every element of $\hat{\phi}^{-1}(y)$ is 0. Then

$$\hat{\psi}^{-1}(y) = \{(y_0, y_1, \ldots, y_{k-1}, 0, y_{k+1}, \ldots), (y_{i-1}, y_i) \subset X\}$$

for $y_0 = y$. The set of points z for which $(0,z) \in X$ is $[0,1]$, so

$$\hat{\psi}^{-1}(y) = \{(y_0, y_1, \ldots, y_{k-1}, 0, z_0, z_1, \ldots) : \\ (y_{i-1}, y_i) \in X,\ (z_{i-1}, z_i) \in X,\ z_0 \in [0,1]\}.$$

Observe that the map $\tau_k(x) = \sigma^k(x)$ is invertible on $\hat{\psi}^{-1}(y)$ and the set $\tau_k(\hat{\psi}^{-1}(y))$ is equal to $\widehat{X}$, so indeed $\hat{\psi}^{-1}(y)$ and $\hat{X}$ are homeomorphic. ∎

We can now prove the theorem revealing the main property of our construction.

THEOREM 4.14. *The set $\widehat{X}$ is a generalized* $\sin(1/x)$*-type continuum.*

Proof. By Lemma 4.13, each fiber $\hat{\psi}^{-1}(y)$ is a singleton or is homeomorphic to $\widehat{X}$. In both cases, it is a connected set, meaning $\hat{\psi}$ is monotone. By Lemma 4.12 and the Baire Category Theorem, the set of degenerate fibers is dense in $\widehat{X}$. It remains to show the approximation property from the definition of generalized $\sin(1/x)$-type continua.

Fix any $y \in [0,1]$ and $\epsilon > 0$. If the preimage of y is degenerate, there is nothing to prove, so assume that $\hat{\psi}^{-1}(y)$ is nondegenerate and fix a subcontinuum $Y_0 \subset \hat{\psi}^{-1}(y)$. By the construction, there is a natural number k such that $\psi_k^{-1}(y)$ is a tranche of X_k. Let π_k be the projection

$$\pi_k \colon \widehat{X} \ni (x_0, x_1, \ldots) \mapsto (x_0, x_1, \ldots, x_k, 0, 0, 0, \ldots) \in X_k.$$

Choose k large enough that $Z_0 = \pi_k(Y_0)$ is a subcontinuum of a tranche of X_k, and any set $V \subset \mathcal{H}$ satisfies

$$d_H(V, \pi_k(V)) < \epsilon/4.$$

By Lemma 4.10, X_k is a generalized $\sin(1/x)$-type continuum, so there is an arc $[a,b] \subset [0,1]$ such that $B = \psi_k^{-1}([a,b])$ approximates Z_0:

$$d_H(B, Z_0) < \epsilon/4.$$

Let A be an extension of B to $\hat{X}$ given by

$$A = \{(x_0, x_1, \ldots, x_k, x_{k+1}, \ldots) \in \hat{X} : (x_1, \ldots, x_k, 0, \ldots) \in B\} = \pi_k^{-1}(B).$$

Such an infinite extension is nonempty, because for any element of B we can consider all replacements of 0 on the $(k+1)$th coordinate using all the elements $(x_k, x_{k+1}) \in X$ and proceed with this process inductively. We can view the process of creation of A as a Cauchy sequence in the hyperspace (or use the fact that π_k is continuous); hence, A is closed. This means that the Hausdorff distance is well defined on A and

$$d_H(A, Y_0) \le d_H(A, B) + d_H(B, Y_0) \le d_H(A, B) + d_H(B, Z_0) + d_H(Z_0, Y_0).$$

To sum up, we get

$$d_H(\hat{\psi}^{-1}([a,b]), Y_0) = d_H(A, Y_0) \le \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon,$$

proving the approximation property, and so $\widehat{X}$ is a generalized $\sin(1/x)$-type continuum. ∎

THEOREM 4.15. *The continuum $\widehat{X}$ does not contain any nondegenerate arcs.*

Proof. Assume on the contrary that there exists a nondegenerate arc A in $\widehat{X}$. If $\hat{\psi}(A) = [a,b]$ for some $a \neq b \in [0,1]$, then by Lemma 4.12 we can find $y \in [a,b]$ with a nondegenerate preimage under $\hat{\psi}$. It follows that $\psi_k^{-1}(y)$ is a tranche in X_k for some arbitrarily large $k \in \mathbb{N}$, and so is the limit set of an oscillatory quasi-arc. But then the projection of A onto X_k is not arcwise connected, which would contradict the fact that A is an arc. This means that

A is a subset of a tranche of $\widehat{X}$ or it does not intersect any tranche, which means it is a singleton.

Since A is nondegenerate, there is $y_0 \in [0,1]$ such that all points $y \in A$ have y_0 in the first coordinate, so they have the form $y = (y_0, x_1, x_2, \ldots)$. By our construction $\sigma(y, x_1, x_2, \ldots) \in \widehat{X}$ and the set $\{(x_1, x_2, \ldots) : (y_0, x_1, x_2, \ldots) \in A\}$ is a nondegenerate arc. By the same argument as before, the second coordinate is the same for all elements of A. By induction, for any $y, \tilde{y} \in A$ and any coordinate i, we have $y_i = \tilde{y}_i$. It follows that A is a singleton, which contradicts our assumptions. We conclude that $\widehat{X}$ contains no nondegenerate arcs. ∎

The well-known example of a nondegenerate continuum that does not contain any arcs is the pseudo-arc (see e.g. [7]), a hereditarily indecomposable continuum.

However, we can prove that $\hat{X}$ has an opposite extreme property, i.e. it is hereditarily decomposable. It turned out that tranched graphs are always decomposable, and in hereditarily tranched graphs the decomposability becomes hereditary by their natural structure.

Proposition 4.16. *Every tranched graph is decomposable; moreover, hereditarily tranched graphs are hereditarily decomposable.*

Proof. Let $\phi\colon X \to Y$ be a continuous map onto a topological graph Y from Definition 3.7. Since $Y = A \cup B$ for two proper subcontinua, $X = \phi^{-1}(A) \cup \phi^{-1}(B)$ showing decomposability. By a similar argument, every subcontinuum which is not completely contained in a tranche is decomposable.

If X is a hereditarily tranched graph, all nondegenerate subcontinua of X are tranched graphs, and so the result follows. ∎

The next result shows that the controlled collapse of a subset of a hereditarily tranched graph leads to another such continuum.

Proposition 4.17. *Let X be a hereditarily tranched graph and $\phi\colon X \to Y$ be an associated mapping. Let $\sim$ be a closed equivalence relation on X with finitely many nondegenerate and connected equivalence classes preserved by ϕ, i.e. if $p \sim q$ then $x \sim y$ for any $x \in \phi^{-1}(\phi(p))$ and $y \in \phi^{-1}(\phi(q))$. Then $X/{\sim}$ is also a hereditarily tranched graph.*

Proof. Define the relation $\wr$ on Y by putting $y_1 \wr y_2$ if and only if there are $x_i \in \phi^{-1}(y_i)$ for $i = 1, 2$ such that $x_1 \sim x_2$. Since $\sim$ is preserved by ϕ, the relation $\wr$ is an equivalence relation. Denote $A = X/{\sim}$ and $B = Y/{\wr}$. First, observe that the map $\psi\colon A \ni [x]_\sim \mapsto [\phi(x)]_\wr \in B$ is well defined and monotone. As $\sim$ is a closed equivalence relation collapsing to a point at most finitely many connected sets in Y, obviously A is a compact metric space and B is a topological graph. As the set of degenerate fibers of ϕ was dense in X, the set of degenerate fibers of ψ is dense in A. The same argument works for a subcontinuum of $X/{\sim}$, so we find that $X/{\sim}$ is a hereditarily tranched graph. ∎

Right now, we have shown that there exists a generalized $\sin(1/x)$-type continua of infinite depth (as defined in Definition 4.18) and width (the number of tranches). By Lemma 3.12 we know that quasi-graphs are hereditarily tranched graphs with finite depth and width (i.e. number of tranches). This, alongside the arcwise connectedness of quasi-graphs, gives four properties that seem to characterize quasi-graphs in the class of tranched graphs. Our goal now is to show that arcwise connected tranched graphs always have finite width. This will remove one necessary condition from the assumptions.

Definition 4.18. For a hereditarily tranched graph X, let

$$\mathrm{lvl}_n(X) = \{T : X = T_0 \supset T_1 \supset \cdots \supset T_{n-1} \supset T_n = T \text{ and } T_{k+1} \text{ is a tranche of } T_k\}.$$

We will call $\sup_{n\in\mathbb{N}} \{n : \mathrm{lvl}_n(X) \neq \emptyset\}$ the *depth* of X.

Notice in particular that if X is a quasi-graph, then the following numbers coincide:

(1) the depth of X considered as a hereditarily tranched graph, and
(2) the maximal order of quasi-arcs of X considered as quasi-graphs.

The following lemma gives a method of reducing complexity of a hereditarily tranched graph, by removing oscillatory quasi-arcs "from the outside in" in such a way that the modified space remains in this class. Notice that if an oscillatory quasi-arc was contained in the limit set of another oscillatory quasi-arc, then it cannot be directly removed since it would make the space no longer closed. This motivates the assumption of not having ancestors in the following lemma, making the procedure of removal of quasi-arcs partially ordered in some sense.

Lemma 4.19. *Let X be a hereditarily tranched graph with a finite set of tranches, and let $L = \varphi([0,\infty)) \subset X$ be an oscillatory quasi-arc without ancestors. Then there is $M \in \mathbb{N}$ such that $X \setminus \varphi((M,\infty))$ is a hereditarily tranched graph. If X is arcwise connected, then $X \setminus \varphi((M,\infty))$ is also arcwise connected.*

Proof. Let $\phi\colon X \to Y$ be the continuous map from Definition 3.7, and L be an oscillatory quasi-arc with $\omega(L) \subset T$ for some tranche $T \subset X$. If L contains a branching point of X or intersects any tranche, we choose M large enough that the quasi-arc $\varphi((M,\infty))$ does not have these properties, which is possible by the following argument. Since L is without ancestors, it maps one-to-one to a topological graph Y, hence can only contain finitely many branching points of X. Hence, there are finitely many "bad" points in L, which we can get rid of by shortening the quasi-arc. Therefore, assume that we have made the above modification when necessary and put $\tilde{L} = \varphi((M, \infty))$. This means that $T \cup \tilde{L}$ is not arcwise connected.

Therefore, if X is arcwise connected then $X_L = X \setminus \varphi((M, \infty))$ is arcwise connected, because an arc connecting any two points in X_L may be taken to avoid intersecting $\tilde{L}$. Let $Y_L = \phi(X \setminus \varphi((M, \infty))$. By our assumptions $\tilde{L} \cap T = \emptyset$, so $\phi(\varphi((M, \infty))$ is an open arc, meaning Y_L is a topological graph. As X was a hereditarily tranched graph with a finite set of tranches, by Lemma 3.15 all of the tranches are unions of limit sets of oscillatory quasi-arcs.

If there exist quasi-arcs $K_1, \dots, K_m \subset X_L$ such that $\bigcup_{i=1}^{m} \omega(K_i) = T$, then X_L is a tranched graph with associated mapping $\phi_L = \phi|_{X_L}$ and $\mathrm{lvl}_1(X) = \mathrm{lvl}_1(X_L)$.

Assume now that for any quasi-arcs $K_1, \dots, K_m \subset X_L$ we have $\bigcup_{i=1}^{m} \omega(K_i) \neq T$. We present the sketch of the following procedure in Figure 9.

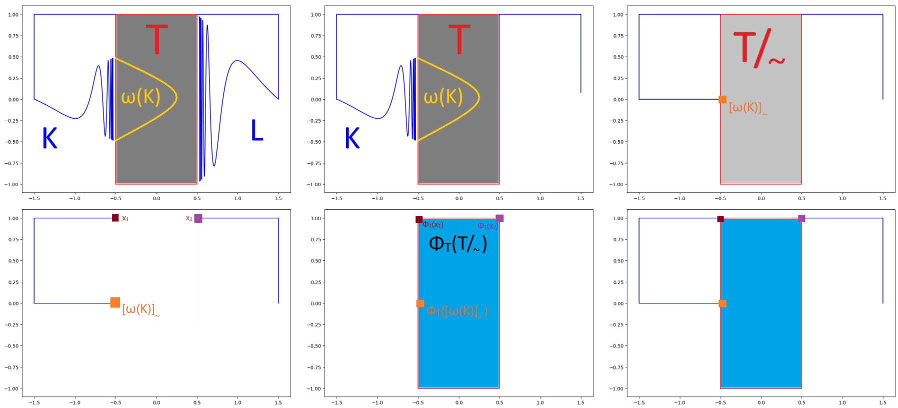

Fig. 9. A sketch of constructions in Lemma 4.19. Top, from left to right: The continua X, X_L and $X_L/{\sim}$. Bottom: X_1 and $\phi_T(T/{\sim})$ with points that will be identified marked with the same color.

From hereditariness of X we know that T is a tranched graph, so let $\eta\colon T \to Z$ be an associated continuous map, where Z is a topological graph. Denote $\Omega = \eta^{-1}(\eta(\omega(X_L) \cap T))$ and let $\sim$ be the equivalence relation such that $a \sim b$ if $a = b$ or there exists a connected component Λ of Ω such that $a, b \in \Lambda$. We extend $\sim$ trivially from T to X_L by adding to it the diagonal in $X_L \times X_L$, so we can view $\sim$ also as a relation on X_L. Clearly the relation $\sim$ is preserved by η, hence, by Proposition 4.17, $T/{\sim}$ is a hereditarily tranched graph. Let ϕ_T be the map from the definition of tranched graph for $T/{\sim}$.

Write X_1 for the closure of $(X_L/\sim)\setminus(T/\sim)$ in $X_L/\sim$ and Y_T for $\phi_T(T/\sim)$. Notice that X_1 has a nonempty intersection with $T/\sim$ and that it is a finite collection of topological graphs. Define an equivalence relation $\approx$ on $X_1 \cup Y_T$ such that $x \approx y$ if and only if (i) $x = y$ or (ii) $x \in T/\sim$, $y \in Y_T$ and $y = \phi_T(x)$ or (iii) $x, y \in T/\sim$ and $\phi_T(x) = \phi_T(y)$. Observe that there are only finitely many equivalence classes of $\approx$ and both X_1 and Y_T are topological graphs, hence $Y_L = (X_1 \cup Y_T)/\approx$ is a topological graph. We define a map $\phi_L\colon X_L \to Y_L$ by putting $\phi_L(x) = (\pi_\approx \circ \pi_\sim)(x)$ for $x \in X_L \setminus T$ and $\phi_L(x) = (\pi_\approx \circ \phi_T \circ \pi_\sim)(x)$ for $x \in T$, where $\pi_\sim$ and $\pi_\approx$ are the natural projections of the relations $\sim$ and $\approx$ respectively.

We claim that all the fibers of ϕ_L are nowhere dense. Consider y such that $\phi_L^{-1}(y)$ is nondegenerate and assume that there is an open set $U \subset \phi_L^{-1}(y) \subset X_L$. This is not possible when $U \cap (X_L \setminus T) \neq \emptyset$ because then ϕ has a fiber which is not nowhere dense. Therefore we may assume that $U \subset T$. But then we must have $U \subset T \setminus \omega(X_L)$ as otherwise it cannot be contained in T by the definition of oscillatory quasi-arc. Thus $U \subset \Omega\setminus\omega(X_L)$, which by the fact that singleton fibers of η are dense in T implies that U must be contained in a fiber of η. But then it is nowhere dense in T, which is again a contradiction, therefore the claim holds. This shows that ϕ_L satisfies the conditions from Definition 3.7, X_L is a hereditarily tranched graph. ∎

In the next lemma we prove that for arcwise connected continua, the image of a tranche has to be contained in a circle. This is in line with the arguments given before.

LEMMA 4.20. *Let X be an arcwise connected tranched graph with an associated mapping $\phi\colon X \to Y$ suppose that the fiber $\phi^{-1}(y)$ is nondegenerate. Then there is a circle $S \subset Y$ such that $y \in S$.*

Proof. Suppose that $y \in Y$ is such that $T = \phi^{-1}(y) \subset X$ is a tranche and there is no circle $S \subset Y$ such that $y \in S$. It is easy to see that y cannot be an endpoint of Y. It follows that $Y \setminus \{y\}$ has at least two connected components.

By Lemma 3.15 the tranche T is a union of limit sets of oscillatory quasi-arcs. Choose an oscillatory quasi-arc L without ancestors such that $\omega(L) \subset T$ and let Y_2 be a connected component of $Y \setminus \{y\}$ that does not intersect $\phi(L)$. Choose $x_1 \in L$ not contained in a tranche and denote $y_1 = \phi(x_1)$. Similarly, there is $y_2 \in Y_2$ such that $\phi^{-1}(y_2)$ is a singleton, so there is a unique $x_2 \in X$ with $y_2 = \phi(x_2)$.

The point y is not an element of a circle, hence any arc with endpoints $\{y_1, y_2\}$ has to contain y. Take any arc $A \subset X$ with endpoints x_1 and x_2, which exists since X is arcwise connected. It follows that $y \in \phi(A)$.

This means that $A \cap (L \cup \omega(L))$ is an arc because otherwise Y contains a circle passing through y. This, combined with the fact that $\omega(L) \cap A \neq \emptyset$, shows that L is not oscillatory, which is a contradiction. ∎

DEFINITION 4.21. We say that a continuum X is *irreducible between a and b* for some $a, b \in X$ if the only subcontinuum containing both a and b is X.

A continuum X irreducible between $x_0, x_1 \in X$ is called a *λ-continuum* if there is a monotone map $\phi\colon X \to [0,1]$ such that $\phi^{-1}(i) = x_i$ for $i \in \{0,1\}$ and every fiber $\phi^{-1}(y)$ is connected.

Suppose X is a λ-continuum. If the union of degenerate fibers is dense in X, then it is a tranched graph. This immediately gives the following.

COROLLARY 4.22. *Suppose X is both an arcwise connected tranched graph and a λ-continuum. Then X is an arc.*

Using Lemma 4.20 we easily obtain the following.

COROLLARY 4.23. *Suppose X is an arcwise connected tranched graph and let $\phi\colon X \to Y$ be the associated mapping. If $a \in \mathrm{End}(Y)$ and $b \in \mathrm{Br}(Y)$ are such that $[a,b] \cap \mathrm{Br}(Y) = \{b\}$, then for all $y \in [a,b)$ the fiber $\phi^{-1}(y)$ is degenerate.*

The following can be used to find an upper bound for the number of tranches.

LEMMA 4.24. *Suppose X is an arcwise connected tranched graph, let $\phi\colon X \to Y$ be the associated mapping, and assume that $y_1 \neq y_2 \in Y$ define tranches. Then there are circles $S_1 \neq S_2 \subset Y$ such that $y_i \in S_i$.*

Proof. Assume on the contrary that there are no circles $S_1 \neq S_2 \subset Y$ such that $y_i \in S_i$. By Lemma 4.20 there is a unique circle S such that $y_1, y_2 \in S$. We claim that S has at least two branching points of X. Suppose not and denote by C_1, C_2 the connected components of $S \setminus \{y_1, y_2\}$. Choose any points $c_i \in C_i$ for $i = 1, 2$. Then every arc connecting them has to pass through y_1 or y_2. This is a contradiction (cf. the proof of Lemma 4.20), so the claim holds.

Fix distinct $b_1, b_2 \subset S \cap \mathrm{Br}(Y)$ and let C_1, C_2 be the connected components of $S \setminus \{b_1, b_2\}$ such that $y_i \in C_i$. The continuum X is arcwise connected, so for points $c_i \in C_i$ there is an arc $A \subset X$ with endpoints in the sets $\phi^{-1}(c_1)$ and $\phi^{-1}(c_2)$. By using the same argument as before, $y_1, y_2 \notin \phi(A)$. But then $S_i = \phi(A) \cup C_i$ are circles such that $y_i \in S_i$ and $S_1 \neq S_2$, which contradicts the uniqueness of S. ∎

Using the standard terminology of algebraic topology, we can state the result as follows: Using Lemma 4.24 inductively, we conclude that for a tranched graph X with associated mapping $\phi\colon X \to Y$ there are at least as many circles in the topological graph Y as there are tranches in X.

Proposition 4.25. *Suppose X is an arcwise connected tranched graph and let $\phi\colon X \to Y$ be the mapping from its definition. Then X has at most $b_1(Y)$ tranches, where $b_1(Y)$ is the first Betti number of the topological graph Y.*

We can use Proposition 4.25 inductively to prove that there are finitely many tranches at any depth of the continuum. Strictly speaking, we have the following.

Lemma 4.26. *Let X be an arcwise connected, hereditarily tranched graph. Then for all $k \in \mathbb{N}$, the set $\operatorname{lvl}_k(X)$ is finite.*

Proof. The Betti number of a graph is always finite; hence, by Proposition 4.25 the set $\operatorname{lvl}_1(X)$ is finite. By Lemma 3.15, this set can be presented as the union of limit sets of (finitely many) oscillatory quasi-arcs in X. Using Lemma 4.19 we remove these quasi-arcs from X, obtaining an arcwise connected continuum X_1 such that $\operatorname{lvl}_1(X_1) = \operatorname{lvl}_2(X)$. Repeating the above argument, we deduce that the set of tranches of X_1 is finite and hence so is the set $\operatorname{lvl}_2(X)$. Continuing this process inductively, we see that for every $N \in \mathbb{N}$ the set $\operatorname{lvl}_N(X)$ is finite, completing the proof. ∎

We can now prove the theorem characterizing those tranched graphs which at the same time satisfy the definition of quasi-graph.

Theorem 4.27. *Let X be a tranched graph. Then X is a quasi-graph if and only if it is an arcwise connected, hereditarily tranched graph of finite depth.*

Proof. Let X be a tranched graph. Assume first that X is a quasi-graph. We immediately see that X is arcwise connected. As we showed in Lemma 3.4 and Theorem 3.16, in this case all tranches are the limit sets of oscillatory quasi-arcs. By Definition 2.3 there are finitely many of them. This means that X has a finite set of tranches and none of them contains an infinite hierarchy of tranches. Finally, Lemma 3.12 shows that X is a hereditarily tranched graph.

Now suppose X is a hereditarily tranched graph which additionally is arcwise connected and of finite depth. Let $\phi\colon X \to Y$ be the map from Definition 3.7. By Lemma 3.15 all tranches are connected unions of limit sets of quasi-arcs. We use Lemma 4.19 inductively to remove all oscillatory quasi-arcs from X, until we get a topological graph, which we denote by G. Suppose we have removed n oscillatory quasi-arcs from X to get G. Let us assume that the indices of oscillatory quasi-arcs we removed are ordered in such a way that L_n is the first quasi-arc we removed and L_1 the last. Then the following hold:

(i) $X = G \cup \bigcup_{j=1}^{n} L_j$ and by our assumption that quasi-arcs have no branching points, $\mathrm{Br}(X) \subset G$ and $\mathrm{End}(X) \subset G$.

(ii) Any oscillatory quasi-arc L_j intersects the topological graph G only at its endpoint.

(iii) In the order used to index the quasi-arcs, for any $i = 1, \dots, n$ the quasi-arcs with lower index $j < i$ were removed from X later than L_i and those with higher index $j > i$ were removed before L_i. This means that $\omega(L_i) \subset G \cup \bigcup_{j=1}^{i-1} L_j$ for any $1 \le i \le n$.

(iv) Suppose now that $\omega(L_i) \cap L_j \neq \emptyset$ for some $i, j \in \mathbb{N}$. Then $j < i$, and so L_i will be removed in the construction leading to G before L_j. If L_j was not a subset of $\omega(L_i)$, we can shorten L_j accordingly obtaining $\omega(L_i) \cap L_j = \emptyset$, so we may assume that $\omega(L_i) \subset L_j$ provided that $\omega(L_i) \cap L_j \neq \emptyset$ for some i, j.

Thus X has all the properties from Definition 2.3, meaning it is a quasi-graph, ending the proof. ■

The following example shows that tranches of a generalized $\sin(1/x)$-continuum are not necessarily generalized $\sin(1/x)$-type continua themselves.

Example 4.28. Let

$$G = (\{-1\} \times [-1/2, 1]) \cup ([-1, 0] \times \{1\}) \cup (\{0\} \times [-1, 1]),$$
$$L_1 = \left\{ \left(x, \tfrac{1}{2}\sin(-\pi/x)(1+x)+1\right) \,\middle|\, x \in [-1, 0) \right\},$$
$$L_2 = \left\{ \left(x, \tfrac{1}{2}\sin(-\pi/x)(1+x)-1\right) \,\middle|\, x \in [-1, 0) \right\},$$
$$X_1 = G \cup L_1 \cup L_2.$$

As X_1 does not satisfy the necessary condition in Theorem 3.16, it is not a generalized $\sin(1/x)$-type continuum (see Figure 10). Denote by φ_1, φ_2 parameterizations of the quasi-arcs L_1 and L_2 respectively. Let $\gamma_N^i \colon [0, 1] \to X_1$

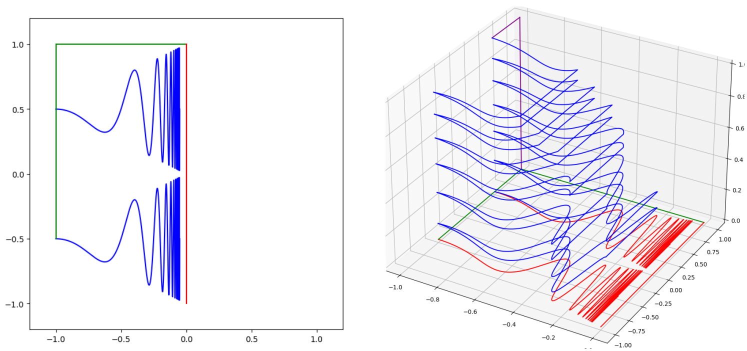

Fig. 10. The quasi-graphs X_1 and X from Example 4.28

for $N \in \mathbb{N}$ be curves defined by

$$\gamma_N^1(t) = \varphi_1(Nt), \quad \gamma_N^2(t) = (1-t)\varphi_1(N) + t\varphi_2(N), \quad \gamma_N^3(t) = \varphi_2(N - Nt),$$
$$\gamma_N^4(t) = \varphi_2(Nt), \quad \gamma_N^5(t) = t\varphi_1(N) + (1-t)\varphi_2(N), \quad \gamma_N^6(t) = \varphi_1(N - Nt).$$

The parameter N decides how far into the quasi-arc L_1 we get, before we move to L_2. Let γ_N be such that $\gamma_N(t) = \gamma_N^i(6t)$ for $t \in \left[\frac{i-1}{6}, \frac{i}{6}\right], i = 1, 2, 3, 4, 5, 6$, and let γ be the curve defined by

$$\gamma(Nt) = \gamma_N(t).$$

Define a continuous map $\varphi_K \colon [0, \infty) \to X_1 \times \mathbb{R}$ by

$$\varphi_K(t) = \left(\gamma(t), \frac{1}{t+1}\right)$$

and denote by A an arc connecting the set $X_1 \times \{0\}$ to the point $\varphi_K(0)$. If we write $K = \varphi_K([0, \infty))$, then

$$X = (X_1 \times \{0\}) \cup K \cup A.$$

By Definition 2.3 the continuum X is a quasi-graph (see Figure 10). By Lemma 3.12 both X and its only tranche X_1 are tranched graphs. However, X_1 is not a generalized $\sin(1/x)$-type continuum.

Thus, we have constructed a generalized $\sin(1/x)$-type continuum that is also a quasi-graph and whose tranche is not a generalized $\sin(1/x)$-type continuum.

All generalized $\sin(1/x)$-type continua are tranched graphs by definition, hence we get the following characterization as a direct consequence of Theorem 4.27.

Corollary 4.29. *Let X be a generalized* $\sin(1/x)$*-type continuum. Then X is a quasi-graph if and only if it is an arcwise connected, hereditarily tranched graph, of finite depth.*

5. Construction of an arcwise connected generalized $\sin(1/x)$-type continuum that is a tranched graph of infinite depth. The aim of this section is to rigorously prove the correctness of the construction advertised in its title (see also Example 5.1). The section can be read independently, so the reader may freely skip the details and continue reading in Section 6.

In Example 4.4 we constructed an infinite depth hereditarily tranched graph, hence the assumption that the continuum is of finite depth is necessary in Theorem 4.27. By the method of construction, this continuum was not arcwise connected. On the other hand, we proved in Lemma 4.26 that in an arcwise connected hereditarily tranched graph on any level (depth) the set of tranches is finite. The next example shows that in these continua

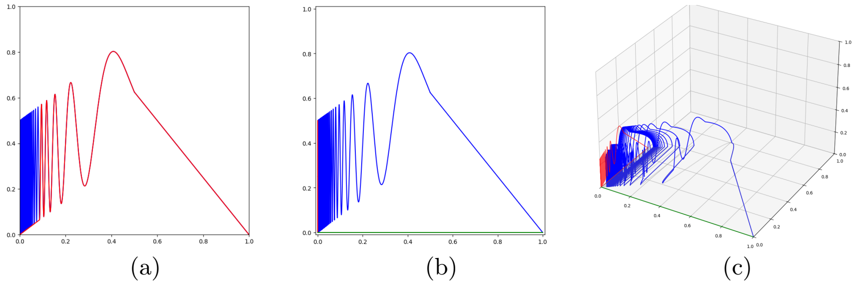

Fig. 11. From left to right: (A) graphs of f_5 (in red) and f (in blue, beyond graph overlap with f_5), (B) the Warsaw circle X and (C) the continuum X_1 from Example 5.1.

infinite depth is possible (the example is even a generalized $\sin(1/x)$-type continuum). In the proof we will use yet another geometrical representation of $\sin(1/x)$-type curves, showing how changing the geometry of the space allows for different constructions, depending on our goals.

Let us informally describe the construction we are going to perform (see Figure 11 for a sketch of the first step). We start with the Warsaw circle X_0 in the plane and we consider the quasi-arc $L_0 \subset X_0$ whose limit is an interval $\omega(L_0)$. Then we modify L_0 to L_1 by incorporating an oscillation in an additional dimension in such a way that now $\omega(L_1) \supset \omega(L_0)$ is a smaller copy of X_0 perpendicular to the original X_0. In this way a continuum X_1 is obtained. This process is repeated inductively, where each time $\omega(L_n)$ is replaced by a smaller copy of X_n and L_n is modified to L_{n+1} which oscillates in one direction more than L_n. As the ultimate step, we prove that the limit continuum X_∞ of the sequence X_n in the Hilbert cube is in fact an arcwise connected generalized $\sin(1/x)$-type continuum. The self-similarity imposed by the construction is then used to show that X_∞ has infinite depth.

Example 5.1. There exists an arcwise connected generalized $\sin(1/x)$-type continuum that is a tranched graph of infinite depth. It will be constructed by induction.

Let us start with the function

$$f(t) = \begin{cases} \frac{1}{4}(\sin\frac{\pi}{t} + 1 + 3t) & \text{if } t \in (0, 1/2], \\ \frac{5}{4} - \frac{5}{4}t & \text{if } t \in [1/2, 1]. \end{cases}$$

Let $\varphi(t) = (t, f(t))$ and $X = ([0,1] \times \{0\}) \cup \overline{L}$ be the Warsaw circle, where $L = \varphi((0,1]))$, i.e. we choose the geometric representation as in Figure 11. Denote by $\{z^{(i)}\}_{i\in\mathbb{N}}$, $\{y^{(i)}\}_{i\in\mathbb{N}}$ the sets of local maxima and minima of f, with ordering $z^{(m)} < z^{(n)}$ if and only if $m > n$. By definition $y^{(1)} = 1$, hence $y^{(n+1)} < z^{(n)} < y^{(n)}$ for every $n \in \mathbb{N}$. For $N > 1$ let

$$f_{N-1}(t) = \begin{cases} f(t) & \text{if } t \in [y^{(N)}, 1], \\ \frac{f(y^{(N)})}{y^{(N)}}t & \text{if } t \in [0, y^{(N)}], \end{cases}$$

and define

(1) $\varphi_0(t) = (t, f(t), 0, \ldots)$, $L_0 = \{\varphi_0(t), t \in (0,1]\} \subset \mathcal{H}$,
(2) $G = [0,1] \times \{0\}^\infty$ and $X_0 = G \cup \overline{L_0}$,
(3) $P^0 = \{P_i^0\}$, where $P_i^0 = [y^{(i+1)}, z^{(i)}]$.

Note that

(4) $f_i|_{P_j^0} = f|_{P_j^0}$ for $j \le i$.

For $i_0 \in \mathbb{N}$ let h_{i_0} be an order-preserving affine homeomorphism from $f(P_{i_0}^0)$ onto $[0,1]$.

LEMMA 5.2. *There is a collection $P^n = \{P^n_{i_0,\ldots i_n}\}_{i_0 \in \mathbb{N},\, i_0 \ge i_1 \ge \cdots \ge i_n}$ of closed intervals for $n = 0, 1, \ldots$ such that for any $n > 0$ and admissible sequence of indices we have $P^n_{i_0,\ldots,i_n} \subset P^{n-1}_{i_0,\ldots,i_{n-1}}$ and $(h_{i_0} \circ f)(P^n_{i_0,\ldots,i_n}) = P^{n-1}_{i_1,\ldots,i_n}$.*

Proof. We will proceed by induction. As defined before, $P^0_{i_0} = [y^{(i_0+1)}, z^{(i_0)}]$ and $P^0 = \{P^0_{i_0}\}_{i_0 \in \mathbb{N}}$. Take any $i_0 \in \mathbb{N}$ and $i_1 \le i_0$. By definition $h_{i_0}(f(P^0_{i_0})) = [0,1]$, hence $P^0_{i_1} \subset (h_{i_0} \circ f)(P^0_{i_0})$, so there is a subinterval of $P^0_{i_0}$ whose image under $(h_{i_0} \circ f)$ is $P^0_{i_1}$. We define $P^1_{i_0,i_1}$ to be this nonempty closed interval.

Suppose we have constructed P^k for all $k \le n$. Choose any natural numbers $i_0 \ge i_1 \ge \cdots \ge i_{n+1}$. By definition $(h_{i_0} \circ f)(P^n_{i_0,\ldots,i_n}) = P^{n-1}_{i_1,\ldots,i_n}$ and there is a closed subinterval $P^n_{i_1,\ldots,i_n,i_{n+1}} \subset P^{n-1}_{i_1,\ldots,i_n}$, that is, $P^n_{i_1,\ldots,i_{n+1}} \subset (h_{i_0} \circ f)(P^n_{i_0,\ldots,i_n})$. Therefore, there is a closed interval $A \subset P^n_{i_0,\ldots,i_n}$ such that $P^n_{i_1,\ldots,i_{n+1}} = (h_{i_0} \circ f)(A)$. We put $P^{n+1}_{i_0,\ldots i_{n+1}} = A$, completing the induction. ∎

In what follows, we assume that the collections P^n of closed intervals (where $n = 0, 1, 2, \ldots$) provided by Lemma 5.2 are fixed.

LEMMA 5.3. *Let $P^n_{i_0,\ldots,i_n} \in P^n$. Then the map*

$$g_{i_0,\ldots i_n} \colon P^n_{i_0,\ldots,i_n} \ni t \mapsto (f_{i_n} \circ h_{i_n} \circ \cdots \circ h_{i_1} \circ f_{i_0} \circ h_{i_0} \circ f)(t) \in [0,1]$$

is well defined and continuous.

Proof. Fix any $t \in P^n_{i_0,\ldots,i_n}$; we will go from the inside out to prove our claim. Note that $f(t) \in f(P^n_{i_0,\ldots,i_n})$, so $(h_{i_0} \circ f)(t) \in P^{n-1}_{i_1,\ldots,i_n} \subset P^0_{i_1}$ and $i_1 \le i_0$. Therefore $f_{i_0}(s) = f(s)$ for $s = (h_{i_0} \circ f)(t)$ by (4). Suppose we know that $(h_{i_k} \circ f_{i_{k-1}} \circ \cdots \circ h_{i_1} \circ f_{i_0} \circ h_{i_0} \circ f)(t) \in P^{n-k}_{i_{n-k},\ldots,i_n}$ for some $k < n$, and set $s = (h_{i_k} \circ \cdots \circ f)(t)$. As $i_{k+1} \le i_k$, by (4) we have $f_{i_k}(s) = f(s)$, so $f_{i_k}(s) = f(s) \in f(P^{n-k}_{i_{n-k},\ldots,i_n}) = h^{-1}_{i_{k+1}}(P^{n-k-1}_{i_{n-k-1},\ldots,i_n})$. Then $(h_{i_{k+1}} \circ f_{i_k} \circ h_{i_k} \circ \cdots \circ f)(t) \in P^{n-k-1}_{i_{n-k-1},\ldots,i_n}$. Completing the induction, $(h_{i_n} \circ f_{i_{n-1}} \circ h_{i_{n-1}} \circ \cdots \circ f)(t) \in P^0_{i_n}$, so the map f_{i_n} is well defined at this point and consequently $g_{i_0,\ldots,i_n}$ is well defined. The map $g_{i_0,\ldots,i_n}$ is continuous as a composition of continuous maps. This finishes the proof. ∎

LEMMA 5.4. *For any $i_0 \geq i_1 \geq \cdots \geq i_n$ we have*

$$g_{i_0,\ldots,i_n}(\mathrm{End}(P^n_{i_0,\ldots,i_n})) = \{0\}.$$

Proof. Fix any $i_0 \geq i_1 \geq \cdots \geq i_n$. The map h_{i_0} is a homeomorphism from $f(P^0_{i_0})$ onto $[0,1]$, and $f_{i_0}(\{0,1\}) = \{0\}$, so g_{i_0} sends the endpoints of $P^0_{i_0}$ to $\{0\}$ for any $i_0 \in \mathbb{N}$. Suppose the result holds for any $k < n$. Choose any $e \in \mathrm{End}(P^n_{i_0,\ldots,i_n})$. By definition $(h_{i_0} \circ f)(P^n_{i_0,\ldots,i_n}) = P^{n-1}_{i_1,\ldots,i_n}$ and $h_{i_0} \circ f$ is a homeomorphism on $P^0_{i_0}$, hence $(h_{i_0} \circ f)(e) = e' \in \mathrm{End}(P^{n-1}_{i_1,\ldots,i_n})$. As $e' \in P^{n-1}_{i_1,\ldots,i_n} \subset P^0_{i_1}$ and $i_1 < i_0$ we find that $f(e') = f_{i_0}(e')$, which implies that

$$\begin{aligned} g_{i_0,\ldots i_n}(e) &= (f_{i_n} \circ h_{i_n} \circ \cdots \circ h_{i_1} \circ f_{i_0} \circ h_{i_0} \circ f)(e) \\ &= (f_{i_n} \circ h_{i_n} \circ \cdots \circ h_{i_1} \circ f_{i_0})(e') \\ &= (f_{i_n} \circ h_{i_n} \circ \cdots \circ h_{i_1} \circ f)(e') \\ &= g_{i_1,\ldots,i_n}(e') = 0. \ \blacksquare \end{aligned}$$

We will use a standard notation $x = (x_0, x_1, \ldots)$ for $x \in \mathcal{H}$. Let $\pi_n\colon \mathcal{H} \ni x \mapsto x_n \in [0,1]$ be the projection onto the $(n+1)$th coordinate,

In the induction process, we use the family P^n, which is fixed before we start the recursive construction. Inductively we will construct (X_n, L_n, φ_n) with the following properties:

- (A1) $X_n = G \cup \overline{L_n} \subset \mathcal{H}$ and $L_n \subset (0,1]^{n+1} \times \{0\}^\infty$,
- (A2) L_n is an oscillatory quasi-arc in X_n with parameterization $\varphi_n\colon (0,1] \to L_n$,
- (A3) $X_n/{\sim}$ is a circle,
- (A4) the unique nondegenerate fiber of $\pi_\sim|_{X_n}$ is $\omega(L_n)$,
- (A5) the map $\pi_0|_{L_n}\colon L_n \to (0,1]$ is a homeomorphism,
- (A6) for every subcontinuum $Y \subset \omega(L_n)$ and every $\epsilon > 0$ there is an arc $[a,b] \in (0,1]$ such that $d_H(Y, \varphi_n([a,b])) < \epsilon$,
- (A7) for $n > 0$ and all $x \in L_n$ we have $(x_0, \ldots, x_n, 0, \ldots) \in L_{n-1}$,
- (A8) for $n > 0$ we have $\omega(L_n) = \frac{1}{2}\theta(X_{n-1})$, where $\frac{1}{2}\theta((x_0, x_1, \ldots)) = (0, \frac{1}{2}x_0, \frac{1}{2}x_1, \ldots)$,
- (A9) for $n > 0$, any natural numbers $i_1 \geq \cdots \geq i_n$ and any $y \in \frac{1}{2}\theta(\varphi_{n-1}(P^{n-1}_{i_1,\ldots i_n}))$ there is a sequence $t^{(i_0)} \in P^n_{i_0,\ldots,i_n}$, $i_0 \geq i_1$, such that $\lim_{i_0\to\infty} \varphi_n(t^{(i_0)}) = y$,
- (A10) for $n > 0$ and any $t \notin \bigcup_{P \in P^n} P$ we have $\pi_{n+1}(\varphi_n(t)) = 0$ and for any $\{i_1, \ldots, i_n\}$ with $i_0 \geq \cdots \geq i_n$ and any $t \in P^n_{i_0,\ldots,i_n}$ we have $\pi_{n+1}(\varphi_n(t)) = \frac{1}{2^n} g_{i_0,\ldots,i_{n-1}}$.

Also, from (A5) and (A7) we find that $\pi_k|_{L_n} \equiv 0$ for every $k > n+1$.

LEMMA 5.5. *The triple* (X_0, L_0, φ_0) *satisfies* (A1)–(A10).

Proof. Since $n = 0$, we only have to check (A1)–(A6). The continuum X_0 is a Warsaw circle defined by (2), hence (A1)–(A4) and (A6) hold. Note that any $x \in L_0$ with $x_0 = t$ is uniquely defined by $(t, f(t), 0, \ldots)$, which gives (A5). ∎

For the general inductive step, suppose we have already defined (X_n, L_n, φ_n) which satisfy (A1)–(A10). We will construct $(X_{n+1}, L_{n+1}, \varphi_{n+1})$ which satisfy (A1)–(A10) as well.

First, for all $t \in (0,1] \setminus \bigcup_{P \in P^n} P$ we define $\varphi_{n+1}(t) = \varphi_n(t)$. Fix any $P = P^n_{i_0,\ldots,i_n}$ and $t \in P$, and put

$$\varphi_{n+1}(t) = \left(t = \pi_0(\varphi_n(t)), \ldots, \pi_{n-1}(\varphi_n(t)), \pi_n(\varphi_n(t)), \frac{1}{2^n} g_{i_0,\ldots,i_n}(t), 0, \ldots\right).$$

By Lemma 5.4 the map φ_{n+1} is continuous. We denote $L_{n+1} = \varphi_{n+1}((0,1])$ and put $X_n = G \cup \overline{L_{n+1}}$. This way the triple $(X_{n+1}, L_{n+1}, \varphi_{n+1})$ is defined.

We see that (A1) holds just by the definition. By definition φ_{n+1} is injective, hence we get (A2). Every $x \in L_{n+1}$ is uniquely determined by $x = \varphi_{n+1}(x_0)$, so in particular (A5) holds.

Choose any $t \in (0,1]$ and let $x = \varphi_{n+1}(t) \in L_{n+1}$. Then we have

$$(x_0, \ldots, x_n, 0, \ldots) = (\pi_0(\varphi_n(t)), \ldots, \pi_{n-1}(\varphi_n(t)), \pi_n(\varphi_n(t)), 0, \ldots) \in L_n,$$

proving (A7). Directly form the definition of the $(n+2)$th coordinate of φ_{n+1}, we get (A10).

As L_{n+1} is oscillatory and $x_0 = 0$ for all $x \in \omega(L_{n+1})$, it is a nondegenerate fiber of X_{n+1}. Furthermore, by definition, $x_0 > 0$ for all $x \in L_{n+1}$. By (A1) for $n+1$ proven above we have $G \cap \{x \in \mathcal{H} : x_0 = 0\} = \{0\}^\infty \subset \omega(L_{n+1})$, hence (A4) holds. As $G \cap \omega(L_{n+1}) = \{0\}^\infty \neq \emptyset$, we see that $X_{n+1}/{\sim}$ is a circle, which proves (A3).

So far, we have shown that $(X_{n+1}, L_{n+1}, \varphi_{n+1})$ satisfies (A1)–(A5), (A7) and (A10). We need to show (A6), (A8) and (A9).

Lemma 5.6. *The triple* $(X_{n+1}, L_{n+1}, \varphi_{n+1})$ *satisfies* (A9).

Proof. First, choose any natural numbers $i_0 \geq i_1 \geq \cdots \geq i_n$ and any $y \in \frac{1}{2}\theta(\varphi_n(P^n_{i_1,\ldots,i_{n+1}}))$. Using (A10) and (A7) for $k < n$ one by one and shifting the coordinates to the right we find that

$$y = \frac{1}{2}\left(0, t, f(t), \ldots, \frac{1}{2^{n-2}} g_{i_1,\ldots,i_{n-1}}(t), \frac{1}{2^{n-1}} g_{i_1,\ldots,i_n}(t), 0, \ldots\right)$$

for some $t \in (0,1]$.

By (A9) there is a sequence $t^{(i_0)} \in P^n_{i_0,\ldots,i_n}$ such that $\lim_{i_0\to\infty} \varphi_n(t^{(i_0)}) = (y_0, \ldots, y_{n+1}, 0, \ldots)$. We wish to show that $\lim_{i_0\to\infty} \varphi_{n+1}(t^{(i_0)}) = y$. By (A7), all coordinates of $\varphi_{n+1}(t^{(i_0)})$ and of $\varphi_n(t^{(i_0)})$ coincide, aside from the $(n+3)$th coordinate. It follows that we only need to check the formula for the $(n+3)$th coordinate. We have

$$\begin{aligned}\lim_{i_0\to\infty} g_{i_0,\dots,i_n}(t^{(i_0)}) &= \lim_{i_0\to\infty}(f_{i_n}\circ h_{i_n}\circ\dots\circ f_{i_0}\circ h_{i_0}\circ f)(t^{(i_0)})\\ &= \lim_{i_0\to\infty}(f_{i_n}\circ h_{i_n})(g_{i_0,\dots,i_{n-1}}(t^{(i_0)}))\\ &= (f_{i_n}\circ h_{i_n})\big(\lim_{i_0\to\infty} g_{i_0,\dots,i_{n-1}}(t^{(i_0)})\big) = \cdots,\end{aligned}$$

but $\pi_{n+1}(\varphi_n(t^{(i_0)})) \to y_{n+1}$. Using (A10) for $\pi_{n+1}(\varphi_n(t^{(i_0)}))$ and $y_{n+1} = \pi_n(\varphi_{n-1}(t))$ we get $g_{i_0,\dots,i_{n-1}}(t^{(i_0)}) \to g_{i_1,\dots,i_{n-1}}(t)$, so we can continue the above chain of equalities:

$$\cdots = (f_{i_n}\circ h_{i_n})(g_{i_1,\dots,i_{n-1}}(t)) = g_{i_1,\dots,i_n}(t) = 2^n y_{n+2},$$

so

$$\lim_{i_0\to\infty}\pi_{n+2}(\varphi_{n+1}(t^{(i_0)})) = \lim_{i_0\to\infty}\frac{1}{2^n}g_{i_0,\dots,i_n}(t^{(i_0)}) = \frac{1}{2^n}2^n y_{n+2} = y_{n+2}.$$

This completes the proof. ∎

LEMMA 5.7. *We have* $\omega(L_{n+1}) = \frac{1}{2}\theta(X_n)$, *so the triple* $(X_{n+1}, L_{n+1}, \varphi_{n+1})$ *satisfies* (A8).

Proof. Choose $\xi \in \frac{1}{2}\theta(X_n)$. Assume first that $\xi = \frac{1}{2}\theta(\varphi_n(t)) \in \frac{1}{2}\theta(L_n)$ for some $t \in (0,1]$. If $\xi \in \frac{1}{2}\theta(\varphi_n(P^n_{i_1,\dots,i_n,i_{n+1}}))$, the result follows from Lemma 5.6. Suppose otherwise. This means that $\varphi_{n-1}(t) = \varphi_n(t)$, so $\xi = \frac{1}{2}\theta(\varphi_{n-1}(t)) \in \frac{1}{2}\theta(L_{n-1}) \subset \frac{1}{2}\theta(X_{n-1})$. By (A8) in the induction hypothesis, there is a sequence $\varphi_n(\xi^{(k)}) \in L_n$ that converges to ξ. If the sequence $\xi^{(k)}$ was chosen to be in the intervals in P^{n+1}, then by Lemma 5.6, ξ would have to be an element of $\frac{1}{2}\theta(\varphi_n(P))$ for some $P \in P^n$, contradicting our assumptions. This means that $\xi^{(k)}$ can be chosen so that $\varphi_n(\xi^{(k)}) = \varphi_{n+1}(\xi^{(k)})$, giving a sequence of points in L_{n+1} converging to ξ.

Now, let $\xi \in \frac{1}{2}\theta(\omega(L_n))$. Choose any $\epsilon > 0$. Then by the definition of the limit set, there is $\xi' \in \frac{1}{2}\theta(L_n)$ with $d(\xi,\xi') < \epsilon$. By the argument above, there is also $\xi'' \in L_{n+1}$ with $d(\xi',\xi'') < \epsilon$, hence $d(\xi,\xi'') < 2\varepsilon$.

Last, assume $\xi \in \frac{1}{2}\theta(G) \subset \frac{1}{2}\theta(X_0)$. Then by (A9) for $n = 1$ there is a sequence $t^{(i)} \in (z^{(i)}, y^{(i)})$ with $\varphi_0(t^{(i)}) \to \xi$. As $\varphi_n(t^{(i)}) = \varphi_0(t^{(i)})$ for every $n > 0$, we get the desired result. ∎

LEMMA 5.8. *For every* $\xi \in \frac{1}{2}\theta(L_n)$ *there is a sequence* $t^{(k)} \in P^0_k$ *such that* $\varphi_{n+1}(t^{(k)}) \to \xi$. *Furthermore, if* $Y \subset \frac{1}{2}\theta(L_n)$ *is a subcontinuum such that* $Y \cap \frac{1}{2}\theta(L_n) \neq \emptyset$, *then there are intervals* $J_k \subset P^0_k$ *such that*

$$\lim_{k\to\infty} d_H(Y, \varphi_{n+1}(J_k)) = 0,$$

and

(1) *if* $Y = \frac{1}{2}\theta(\varphi_n([a,b])$ *for some* $0 < a < b \leq 1$, *then*

$$J_k = [(f|_{P_k^0})^{-1}(a), (f|_{P_k^0})^{-1}(b)];$$

(2) *if* $Y = \frac{1}{2}\theta(\overline{\varphi_n((0,b])})$ *for some* $b \in (0,1]$, *then*

$$J_k = [y^{(k+1)}, (f|_{P_k^0})^{-1}(b)].$$

Proof. In principle, the result follows from the use of (A9) for $k < n$, considering the points whose $(k+1)$th coordinate is zero. We can also take any $P_{i_0}^0$ to get

$$\varphi_{n+1}(P_{i_0}^0) = \left\{ \left(t, f(t), \frac{1}{2} g_{i_0}(t), \ldots, \frac{1}{2^n} g_{i_0,\ldots,i_n}(t), 0, \ldots \right) : t \in P_{i_0}^0 \right\}.$$

But $h_{i_0} \circ f$ is a homeomorphism onto $P_{i_0}^0$ and $(h_{i_0} \circ f)(P_{i_0,\ldots,i_n}^n) \subset P_{i_1}^0$ with $i_1 \leq i_0$, so $f(s) = f_{i_1}(s)$ for all $s \in (h_{i_0} \circ f)(P_{i_0,\ldots,i_n}^n)$. Therefore, we can write

$$\begin{aligned} &\varphi_{n+1}(P_{i_0}^0) \\ &= \left\{ \left(f^{-1}|_{P_{i_0}^0}(y), y, \frac{1}{2} f(h_{i_0}(y)), \ldots, \frac{1}{2^n} g_{i_1,\ldots,i_n}(h_{i_0}(y)), 0, \ldots \right) : y \in f(P_{i_0}^0) \right\}. \end{aligned}$$

Fix any $\xi \in \frac{1}{2}\theta(L_n)$, say $\xi = \frac{1}{2}\big(0, 2y, f(2y), \ldots, \frac{1}{2^{n-1}} g_{i_1,\ldots,i_n}(2y), 0, \ldots\big)$ for some $y \in (0, 1/2]$.

Note that $f(P_i^0) = [f(y^{(i+1)}), f(z^{(i)})]$ with $f(y^{(i+1)})$ decreasing to 0 and $f(z^{(i)})$ decreasing to $1/2$. For large i there is $t^{(i)} \in P_i^0$ such that $f(t^{(i)}) = y$. Since h_i is an affine homeomorphism for any i, it is also clear that $\lim_{i\to\infty} h_i(y) = 2y$. This proves that $\varphi_{n+1}(t^{(i)}) \to \xi$.

The proof of the "furthermore" part is analogous. Simply, for (1) it is enough to prove convergence at the endpoints of A and extend it to all other points with coordinates in $\pi_0(J_k)$. The details are left to the reader. For (2) notice that $\varphi_{n+1}(y^{(k+1)}) = (y^{(k+1)}, f(y^{(k+1)}), 0, \ldots)$, but as mentioned above, $y^{(k+1)} \to 0$, so $\lim_{k\to\infty} \varphi_{n+1}(y^{(k+1)}) = 0^\infty \in \frac{1}{2}\theta(\omega(L_n))$, but J_k are connected, meaning so is $\lim_{k\to\infty} J_k$. That along with the fact that $\lim_{k\to\infty}(J_k) \cap \frac{1}{2}\theta(L_n) \neq \emptyset$ gives the result. ∎

LEMMA 5.9. *We have*

$$\lim_{k\to\infty} d_H\big(\varphi_{n+1}([z^{(k+1)}, y^{(k)}]), \tfrac{1}{2}\theta(G)\big) = 0.$$

Proof. As $[z^{(k+1)}, y^{(k)}]$ intersects the intervals of P^0 only at the endpoints, where we keep the zero coordinates, we have $\varphi_{n+1}([z^{(k+1)}, y^{(k)}]) = \varphi_0([z^{(k+1)}, y^{(k)}])$. The result follows from the fact that $\lim_{k\to\infty} f(y^{(k)}) = 0$ and $\lim_{k\to\infty} f(z^{(k)}) = \frac{1}{2}$, and $\frac{1}{2}\theta(G) = \{0\} \times [0, 1/2] \times \{0\}^\infty$. ∎

LEMMA 5.10. *The triple* $(X_{n+1}, L_{n+1}, \varphi_{n+1})$ *satisfies* (A6).

Proof. Fix any nondegenerate subcontinuum $Y \subset \omega(L_{n+1})$.

Assume first that $Y \subset \frac{1}{2}\theta(L_n)$; then Y is an arc. By Lemma 5.8 there is a sequence $[a^{(k)}, b^{(k)}] \subset P^0_k$ such that $\varphi_{n+1}([a^{(k)}, b^{(k)}]) \to Y$ in the Hausdorff metric.

Assume now that $Y \subset \frac{1}{2}\theta(\overline{L_n})$ and fix $\epsilon > 0$. By the induction hypothesis (A5), we can find $[a, b] \subset (0, 1]$ such that $d(\frac{1}{2}\theta(\varphi_n([a, b])), Y) < \epsilon$. Then, by the step above, we can find $[a', b'] \subset (0, 1]$ such that

$$d_H\big(\varphi_{n+1}([a', b']), \tfrac{1}{2}\theta(\varphi_n([a, b]))\big) < \epsilon,$$

so $d_H(\varphi_{n+1}([a', b']), Y) < 2\epsilon$.

Suppose $Y \subset \frac{1}{2}\theta(G)$ and fix $\epsilon > 0$. Choose $[a^{(k)}, b^{(k)}] \subset (z^{(k)}, y^{(k)})$ with $\varphi_0([a^{(k)}, b^{(k)}]) \to Y$. Since $(z^{(k)}, y^{(k)}) \cap \bigcup_{P \in P^0} P = \emptyset$, we find that $\varphi_{n+1}([a^{(k)}, b^{(k)}]) = \varphi_0([a^{(k)}, b^{(k)}])$, giving the desired result.

We are left to consider the case $\operatorname{int}(Y_1) = \operatorname{int}(Y \cap \frac{1}{2}\theta(G)) \neq \emptyset$ and $\operatorname{int} Y_2 = \operatorname{int}(Y \cap \frac{1}{2}\theta(\overline{L_n})) \neq \emptyset$. If $Y = \omega(L_{n+1})$, then the statement is obvious by the definition of limit set. Consider first the case where $Y_1 \cap Y_2$ is one point, either $e^0 = (0, 0, \ldots)$ or $e^1 = (0, \frac{1}{2}, 0, \ldots)$ (the endpoints of $\frac{1}{2}\theta(G)$).

In the case of e^0, fix $\delta^{(k)}_1, \delta^{(k)}_2$ such that $\varphi_{n+1}([y^{(k+1)}, z^{(k)} - \delta^{(k)}_2]) \to Y_1$ and $\varphi_{n+1}([z^{(k+1)} + \delta^{(k)}_1, y^{(k+1)}]) \to Y_2$, using Lemma 5.8 .

This gives $\varphi_{n+1}([z^{(k+1)} + \delta^{(k)}_1, z^{(k)} - \delta^{(k)}_2]) \to Y$. As $e_0 \in Y$, by connectedness we have $\omega(L_{n+1}) \subset Y$, hence $\lim_{k\to\infty} \varphi_{n+1}(y^{(k)}) \in Y$ and so the arc approximating Y_1 can be chosen to include $y^{(k+1)}$.

For e_1 we act analogously to get $\varphi_{n+1}([y^{(k+1)} + \delta_1, y^{(k)} - \delta_2]) \to Y$.

Finally, consider the case where $Y_1 \cap Y_2$ contains both e^0, e^1. Then either $Y_1 = \frac{1}{2}\theta(\overline{L_n})$ or $Y_2 = \frac{1}{2}\theta(G)$ because Y is connected. Without loss of generality suppose $\frac{1}{2}\theta(G) \subset Y$. It follows that $Y_1 = \frac{1}{2}\theta(\overline{L_n}) \setminus \frac{1}{2}\theta(\varphi_n([a, b]))$ for some $0 < a < b \leq 1$. Denote $a^{(k)} = (f|_{P^0_k})^{-1}(a)$ and $b^{(k)} = (f|_{P^0_k})^{-1}(b)$. By Lemma 5.8 we have $\varphi_{n+1}([b^{(k+1)}, z^{(k+1)}] \cup [y^{(k)}, a^{(k)}]) \to Y_1$ and by Lemma 5.9 we obtain $\varphi_{n+1}([z^{(k+1)}, y^{(k)}]) \to \frac{1}{2}\theta(G) = Y_2$. Altogether this gives $\varphi_{n+1}([b^{(k+1)}, a^{(k)}]) \to Y$. ∎

Observe that the maps φ_n form a Cauchy sequence (in the space of continuous maps $[0, 1] \to \mathcal{H}$) and therefore $\varphi_\infty(t) = \lim_{n\to\infty} \varphi_n(t)$ is a well-defined continuous map.

Note that $\pi_0(\varphi_\infty(t)) = t$, hence is 1-1. This shows that $L_\infty = \varphi_\infty((0, 1])$ is an oscillatory quasi-arc in X_∞.

We have $X_\infty \cap \{x \in \mathcal{H} : x_0 = 0\} = \omega(L_\infty)$, so $X_\infty/{\sim}$ is a circle, and the map $\phi = \pi_\sim|_{X_\infty}$ has a unique tranche $\omega(L_\infty)$, so X_∞ is a tranched graph.

Similarly to Example 4.4, the continuum X_∞ has a self-similar property, i.e. $\omega(L_\infty) = \frac{1}{2}\theta(X_\infty)$, meaning L_∞ is an ∞-order oscillatory quasi-arc.

Lemma 5.11. *X_∞ is a generalized $\sin(1/x)$-type continuum.*

Proof. Choose any continuum $Y \subset \omega(L_\infty)$ and any $\epsilon > 0$. Pick n large enough that for any $x, y \in \mathcal{H}$ if $x_i = y_i$ for all $i \leq n$ then $d(x, y) < \epsilon$. Then for any subcontinuum $C \subset \mathcal{H}$ and $C_n = \{(x_0, \dots, x_n, 0, \dots) \in \mathcal{H} : x \in C\}$ we have $d_H(C, C_n) < \epsilon$.

It follows that $d_H(X_n, X_\infty) < \epsilon$. Let Y_n be the projection of Y. It follows that $d_H(Y, Y_n) < \epsilon$. As X_n is a generalized $\sin(1/x)$-type continuum, there is an arc $[a, b] \subset (0, 1]$ with $d_H(Y_n, \varphi_n([a, b])) < \epsilon$. By the choice of ϵ we have $d_H(\varphi_n([a, b]), \varphi_\infty([a, b])) < \epsilon$. Thus, the triangle inequality gives $d_H(\varphi_\infty([a, b]), Y) < 3\epsilon$. ∎

Lemma 5.12. *The continuum X_∞ is arcwise connected.*

Proof. It is enough to show that for every $x \in X_\infty$ there is an arc from x to $(1, 0, 0, \dots) = \varphi_\infty(1) \in X_\infty$. If $x \in G$, then the arc is $[x_0, 1] \times \{0\}^\infty$; if $x \in L_\infty$, then we take $\varphi_\infty([x_0, 1])$. Suppose now $x_0 = 0$, but $x_n \neq 0$ for some n. Then $x \in \frac{1}{2^n}\theta^n(L_\infty \cup G)$, so the desired arc is $A = G \cup \frac{1}{2}\theta(G) \cup \dots \cup \frac{1}{2^n}\theta^n(G \cup \varphi_\infty([x_n, 1]))$ or $A = G \cup \frac{1}{2}\theta(G) \cup \dots \cup \frac{1}{2^n}\theta^n([x_0, 1] \times \{0\}^\infty)$ depending on the location of x. ∎

6. Relation to other classes and dynamics of tranched graph maps. In [14] M. Mihoková studied minimal sets on continua with a free interval, where a *free interval* is any space homeomorphic to $\mathbb{R}$. We say that J is a dense free interval in X if J is a free interval and $\overline{J} = X$. In [14] the objects studied are continua X that can be expressed in the form

$$X = L \cup J \cup R$$

where L, R are nowhere dense locally connected continua, disjoint from the dense free interval J that can be split into two rays J_L and J_R such that $\omega(J_L) = L$ and $\omega(J_R) = R$. Similarly to [14], we will denote by $\mathcal{C}$ the class of such continua X.

Lemma 6.1. *Suppose $X \in \mathcal{C}$. Then X is a tranched graph.*

Proof. Let $Y = X/\!\sim$ where $x \sim y$ if and only if $x = y$ or $x, y \in L$ or $x, y \in R$. Let $\phi\colon X \to Y$ be the associated quotient map. Since L and R are closed and nowhere dense, ϕ is continuous and monotone, and moreover all fibers of ϕ are nowhere dense and the only (possible) nondegenerate fibers of ϕ are L and R. It follows that the set of points with degenerate preimage is dense, so ϕ satisfies Definition 3.7 and hence X satisfying the definition from [14] is a tranched graph. ∎

Using [14, Remark 5] it is possible to classify all graphs Y that can appear in the definition of the tranched graphs in this class:

Remark 6.2. Let $X = L \cup J \cup R$, and let $\phi\colon X \to Y$ be a mapping from the definition of tranched graph. Then, up to homeomorphism, the following hold:

(a) If both L and R are singleton then X has no tranches and

- if $L = R$, then $X = Y$ is a circle,
- if $L \neq R$, then $X = Y = [0, 1]$.

(b) If L is nondegenerate and R is a singleton, then L is the only tranche of X and

- if $L \cap R = \emptyset$, then $Y = [0, 1]$ and $\phi^{-1}(0) = L$ is a tranche,
- if $L \cap R \neq \emptyset$, then Y is a circle.

(c) if both L and R are nondegenerate, then

- if $L \cap R = \emptyset$, then $Y = [0, 1]$ and X has two tranches: $\phi^{-1}(0) = L$ and $\phi^{-1}(1) = R$,
- if $L \cap R \neq \emptyset$, then Y is a circle and X has one tranche $L \cup R$.

It is easy to check that all examples provided by [14] satisfy the definition of a generalized $\sin(1/x)$-type continuum. The topological structure of continua with dense free interval is somewhat restricted; still, analyzing the examples provided earlier in this paper, it is not hard to show that the elements of the class $\mathcal{C}$ need not be generalized $\sin(1/x)$-type continua, nor be hereditarily tranched graphs.

We say that a map $f\colon X \to X$ is (*topologically*) *mixing* if for any nonempty open subsets $U, V \subset X$ there is $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$. In the standard hierarchy of chaotic maps, the definition of mixing maps can be extended to a stronger definition of topologically exact maps, where a map $f\colon X \to X$ is (*topologically*) *exact* or *locally eventually onto* (leo) if for every open set $U \subset X$, there exists $N \in \mathbb{N}$ such that $f^N(U) = X$.

In the third chapter of his doctoral thesis ([3], in Polish) Drwiega studied the dynamics of the $\sin(1/x)$-curve, which can be viewed in the framework of this paper as a simple generalized $\sin(1/x)$-type continuum. He showed a lower bound on the topological entropy of a continuous topologically transitive map of a $\sin(1/x)$-continuum, which is $\log 3$. He also proved that for a space consisting of two quasi-arcs accumulating on an interval, no mixing map is topologically exact – a result we extend in Theorem 6.7 by a different argument. In a more general setting, the dynamics of quasi-graph maps was much more studied (see e.g. [12, 11, 21]) than in the case of generalized $\sin(1/x)$-type continua (see [6] for some comments). The first result that is inspired by the above studies is that the set of tranches is invariant under any continuous onto map. This property imposes essential restrictions on topological and ergodic properties of dynamical systems on these spaces.

Lemma 6.3. *Suppose X is a tranched graph with a finite set of tranches and let $\phi\colon X \to Y$ be an associated map. Then for any continuous surjective* *mapping $f\colon X \to X$, the set T_X of tranches of X, i.e.*

$$T_X = \{x \in X : \phi^{-1}(\phi(x)) \neq \{x\}\},$$

is f-invariant, meaning $f(T_X) = T_X$.

Proof. By Lemma 3.15, all tranches of X are limit sets of oscillatory quasi-arcs.

Suppose first that there is $x \in T_X$ such that $f(x) \notin T_X$. Denote by $L_1, \dots, L_k$ the set of quasi-arcs and assume that $x \in \omega(L_i)$. Suppose that there is $\tilde{x} \in \omega(L_i)$ with $f(\tilde{x}) \neq f(x)$. By continuity, we find that $f(L_i)$ is nondegenerate and arcwise connected, hence contains an oscillatory quasi-arc and $f(\omega(L_i))$ is a nondegenerate limit set. This means that $f(\omega(L_i)) \subset T_X$ and so $f(x) \in T_X$, contradicting the assumptions. Therefore $\{f(x)\} = f(\omega(L_i))$ for any oscillatory quasi-arcs L_i such that $x \in \omega(L_i)$, in particular $f(\omega(L_i)) \cap T_X = \emptyset$. This means that $f(L_i \cup \omega(L_i))$ is a topological graph and so $f(L_i)$ does not contain an oscillatory quasi-arc. This implies that at least one oscillatory quasi-arc in X does not map onto an oscillatory quasi-arc. But any oscillatory quasi-arc must be an image of an oscillatory quasi-arc, which would contradict surjectivity. This shows that $f(T_X) \subset T_X$.

Suppose now that there is $y \in T_X$ such that $y \neq f(x)$ for all $x \in T_X$. This means there is an oscillatory quasi-arc $L \subset X$ that no oscillatory quasi-arc maps to, because otherwise, if $f(K) = L$, then $f(\omega(K)) = \omega(L)$, and as a consequence there is $x \in \omega(K)$ such that $f(x) = y$. But f is surjective, leading to a contradiction. ∎

Notice that in general we cannot say that the set $X \setminus T_X$ is also invariant, so Lemma 6.3 cannot be extended beyond T_X.

The following lemma will be an important tool in describing possible topological dynamics on tranched graphs.

Lemma 6.4. *Let X be a tranched graph. Then X is a Peano continuum if and only if it is a topological graph.*

Proof. Let $\phi\colon X \to Y$ be the associated continuous monotone map. Assume that X is a Peano continuum but is not a topological graph. It follows that there is at least one tranche $T = \phi^{-1}(y)$ for some $y \in Y$. Denote by n the valence of $y \in Y$. As T is nondegenerate, we can choose $n+1$ distinct points $\{x_0, \dots, x_n\} \subset T$, denote $\delta = \max d(x_i, x_j)/2$ and let $U_i \subset B(x_i, \delta)$ be open connected sets. Denote $S = \phi(\bigcup_{i=1}^n U_i)$ and observe that S contains an s-star centered at y for some $s \leq n$, but does not contain k-stars for $k > n$. In particular, there is an arc $A \subset Y$ such that $A \subset \phi(U_i) \cap \phi(U_j), j \neq i$. By the density of singleton fibers, there is $y_0 \in A$ with a degenerate preimage. But U_i and U_j are disjoint, hence the fiber $\phi^{-1}(y_0)$ cannot intersect both, leading to a contradiction. ∎

Lemma 3.1 and Corollary 3.2 in [12] state that for a quasi-graph self-map, the image of a topological graph does not contain any oscillatory quasi-arc. The continuous image of a Peano continuum is a Peano continuum as well; therefore, Lemma 6.4 extends the result of [12] to a more general setting of arcwise connected tranched graphs.

COROLLARY 6.5. *Let X be an arcwise connected tranched graph and let $G \subset X$ be a topological graph. For any continuous map $f\colon X \to X$, the set $f(G)$ does not contain any oscillatory quasi-arcs. In particular, if $f(G)$ is nondegenerate, then $f(G)$ is a topological graph.*

Now we are ready to show that tranched graphs may support complicated dynamics.

If we glue two Warsaw circles along their tranches (Figure 12) we get an arcwise connected tranched graph, but whose set of singleton fibers is not arcwise connected. It is easy to verify that such a continuum does not admit a mixing map. On the contrary, the double sided $\sin(1/x)$-continuum (Figure 8) is not arcwise connected, but has arcwise connected set of singleton fibers.

THEOREM 6.6. *Suppose that X is a tranched graph with associated map $\phi : X \to Y$, where Y is a topological graph. Assume that the set $\{x : \phi^{-1}(\phi(x)) = \{x\}\}$ is arcwise connected. Then there exists a topologically mixing map $f\colon X \to X$.*

Proof. Removing the images of tranches from Y keeps the space arcwise connected, therefore the number of tranches for X is bounded from above by $b_1(Y)$ (the disconnecting number of Y) and consequently the continuum X has finitely many tranches. Denote $N = \{y \in Y : \phi^{-1}(y)$ is nondegenerate$\} = \{y_1, \dots, y_n\}$. Let Y_1 be obtained by compactifying $Z_1 = Y \setminus \{y_1\}$ in such a way that $A_1 = Y_1 \setminus Z_1 \subset \mathrm{End}(Y_1)$. Informally this means that we "cut" the graph Y at y_1 obtaining $\mathrm{val}(y_1)$ endpoints. Our assumptions guarantee that Z_1 and Y_1 are connected.

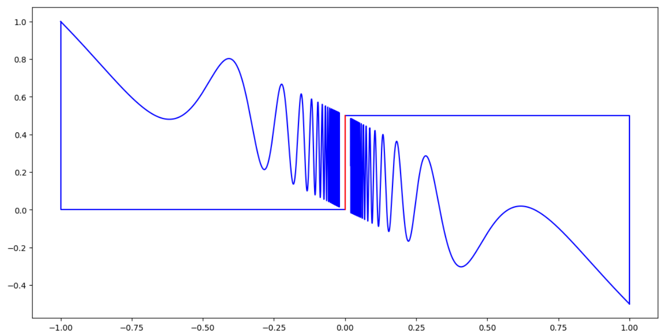

Fig. 12. An arcwise connected tranched graph that does not admit a mixing map.

Continue this construction recursively denoting by Y_k the graph obtained by compactifying $Z_k = Y_{k-1} \setminus \{y_k\}$ in such a way that $A_k = Y_k \setminus Z_k \subset \mathrm{End}(Y_k)$. Denote by $\sim$ the equivalence relation identifying points in each set A_i, i.e. $a \sim b$ if $a = b$ or $a, b \in A_i$ for some i. Note that $Y_n/{\sim} = Y$ up to homeomorphism. By [5], there is a pure mixing map $g\colon Y_n \to Y_n$ with $\mathrm{End}(Y_n) \subset \mathrm{Fix}(g)$. Moreover, using techniques from the construction of a purely mixing map on the interval (see for example [20, Chapter 2.2]), we can deduce that fixed points of g accumulate on the set of endpoints of Y_N in any prescribed way (i.e. we can control the speed of convergence in the construction). We can pull this map back to Y, by setting $h(y) = g(y)$ on points outside A_i and $h(y) = y$ on points in N. Finally, we can define the map $f\colon X \to X$ to be $f(\phi^{-1}(y)) = \phi^{-1}(h(y))$ if y is the image of a degenerate fiber and $f(x) = x$ if x is an element of a tranche of X.

The map is well defined and monotone, and since we may control the convergence of fixed points when defining g, we can easily see that f is also continuous. As g was topologically mixing, so is h, and as a result so is f. ∎

Recall that a map $f\colon X \to X$ is topologically exact if for every open set $U \subset X$, there exists $N \in \mathbb{N}$ such that $f^N(U) = X$. These maps are to some extent excluded on tranched graphs, except for very regular ones, as shown below.

Theorem 6.7. *Suppose that X is a tranched graph with a finite and nonempty set of tranches. Then X does not admit a topologically exact map.*

Proof. Suppose $f\colon X \to X$ is topologically exact. As the set of tranches of X is finite, there is an open connected set that is mapped injectively to a topological graph. Hence, there is a topological graph $G \subset X$. As f is topologically exact, there is a natural number n for which $f^n(G) = X$. As G is a topological graph, X is a continuous image of a Peano continuum, so it is a Peano continuum. This contradicts Lemma 6.4. ∎

The assumption that the continuum has finitely many tranches in Theorem 6.7 is used to generate a Peano subcontinuum with nonempty interior. If the set of tranches is infinite, this argument cannot be used anymore. The following example shows that for complicated generalized $\sin(1/x)$-type continua, we can get complex dynamics both in a global (topologically exact map) and local (infinite topological entropy) sense.

Example 6.8. *Let $\widehat{X}$ be a continuum constructed in Example* 4.8 *and $\sigma\colon \widehat{X} \to \widehat{X}$ be the left shift, defined as in Lemma* 4.9. *Then σ is topologically exact and has infinite topological entropy.*

Proof. By our construction, if $(x_0, x_1, \ldots) \in \widehat{X}$, then $\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots) \in \widehat{X}$, so map σ is well defined on $\widehat{X}$. Now choose an open set

$U \subset \widehat{X}$. By Lemma 4.12, the set of tranches of $\widehat{X}$ is dense in $\widehat{X}$, so there is a tranche $T = \hat{\phi}^{-1}(y) \subset U$. By Lemma 4.13, T is homeomorphic to $\widehat{X}$ and for some $n \in \mathbb{N}$ we have $\sigma^n(T) = \widehat{X}$. As $T \subset U$ it follows that $\sigma^n(U) = \widehat{X}$, so the map $\sigma\colon \widehat{X} \to \widehat{X}$ is topologically exact.

Pick two tranches T_1, T_2 of $\widehat{X}$. By Lemma 4.13 they are homeomorphic to $\widehat{X}$, so for any sequence $s \in \{0,1\}^{\mathbb{N}}$ there is a point $x_s \in \widehat{X}$ such that $f^i(x) \in T_{s(i)}$ for all $i \in \mathbb{N}$. As the topological entropy of the full shift on two symbols is $\log 2$, we infer that $h_{\mathrm{top}}(\sigma) \geq \log 2$. We can repeat this reasoning by picking k tranches to get $h_{\mathrm{top}}(\sigma) \geq \log k$. Freedom of choice of k gives $h_{\mathrm{top}}(\sigma) = \infty$. ∎

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